In Chapter 3, you solved systems of equations.

In Chapter 4, you will:
- Organize data in matrices.
- Perform operations with matrices and determinants.
- Find inverses of matrices.
- Use matrices to solve systems of equations.

GRAPHICS Computer graphics and animation use complex models for characters, objects, and scenery. These computer models describe the shapes of objects and the motions of characters. The animation models use matrices to describe the locations of specific points in the images.
Diagnose Readiness | You have two options for checking prerequisite skills.

1 Textbook Option  Take the Quick Check below. Refer to the Quick Review for help.

<table>
<thead>
<tr>
<th>QuickCheck</th>
<th>QuickReview</th>
</tr>
</thead>
</table>
| **Example 1** Name the additive inverse and the multiplicative inverse for each number. (Lesson 1-2) ** \[
1. 4 \quad 2. -15 \\
3. 0.2 \quad 4. -1.35 \\
5. -\frac{3}{4} \quad 6. 2\frac{1}{3}
\]**

**Example 2** Simplify each expression. (Lesson 1-2) ** \[
7. 6(x + 2y) \\
8. 4(x + 5) - 3 \\
9. -4(3x) - (7x - 6) \\
10. 5(2x - 5) - \frac{1}{3}(4x + 1) \\
11. 6(2x - 1) - 3(y - x) + 0.5(4x - 6)
\]**

**Example 3** Solve each system of equations by using either substitution or elimination. (Lesson 3-2) ** \[
12. y = x + 3 \quad 13. 2x - 5y = -18 \\
\quad 2x - y = -1 \quad \quad 3x + 4y = 19 \\
14. 4y + 6x = -6 \quad 15. x = y - 8 \\
\quad 5y - x = 35 \quad \quad 4x + 2y = 4
\]**

**MONEY** The student council paid $15 per registration for a conference. They also paid $10 for T-shirts for a total of $180. Last year, they spent $12 per registration and $9 per T-shirt for a total of $150 to buy the same number of registrations and T-shirts. Write and solve a system of two equations that represents the number of registrations and T-shirts bought each year.

2 Online Option  Take an online self-check Chapter Readiness Quiz at connectED.mcgraw-hill.com.
You will learn several new concepts, skills, and vocabulary terms as you study Chapter 4. To get ready, identify important terms and organize your resources. You may wish to refer to Chapter 0 to review prerequisite skills.

### Foldables Study Organizer

**Matrices** Make this Foldable to help you organize your Chapter 4 notes about matrices. Begin with one sheet of 11” by 17” paper.

1. **Fold** 2” tabs on each of the short sides.
2. **Fold** in half in both directions. Open and cut as shown.
3. **Refold** along the width. Staple each pocket.
4. **Label** pockets as Operations, Transformations, Determinants/Cramer’s Rule, and Inverses/Systems. Place index cards for notes in each pocket.

### New Vocabulary

<table>
<thead>
<tr>
<th>English</th>
<th>Español</th>
</tr>
</thead>
<tbody>
<tr>
<td>element</td>
<td>elemento</td>
</tr>
<tr>
<td>dimensions</td>
<td>tamaño</td>
</tr>
<tr>
<td>row matrix</td>
<td>matriz fila</td>
</tr>
<tr>
<td>column matrix</td>
<td>matriz columna</td>
</tr>
<tr>
<td>square matrix</td>
<td>matriz cuadrada</td>
</tr>
<tr>
<td>zero matrix</td>
<td>matriz nula</td>
</tr>
<tr>
<td>equal matrices</td>
<td>matrices iguales</td>
</tr>
<tr>
<td>scalar</td>
<td>escalar</td>
</tr>
<tr>
<td>vertex matrix</td>
<td>matriz de vértice</td>
</tr>
<tr>
<td>preimage</td>
<td>preimagen</td>
</tr>
<tr>
<td>image</td>
<td>imagen</td>
</tr>
<tr>
<td>rotation</td>
<td>rotación</td>
</tr>
<tr>
<td>determinant</td>
<td>determinante</td>
</tr>
<tr>
<td>Cramer’s Rule</td>
<td>regula de Crámer</td>
</tr>
<tr>
<td>coefficient matrix</td>
<td>matriz coeficiente</td>
</tr>
<tr>
<td>identity matrix</td>
<td>matriz identidad</td>
</tr>
<tr>
<td>inverse matrix</td>
<td>matriz inversa</td>
</tr>
<tr>
<td>matrix equation</td>
<td>ecuación matriz</td>
</tr>
<tr>
<td>variable matrix</td>
<td>matriz variables</td>
</tr>
<tr>
<td>constant matrix</td>
<td>matriz constante</td>
</tr>
</tbody>
</table>

### Review Vocabulary

- **coordinate plane**: Algebra 1/plan de coordenadas: the plane connecting the x- and y-axes
- **system of equations**: p. 135/sistema de ecuaciones: a set of equations with the same variables

---

*Images and diagrams are not included in the text.*
Introduction to Matrices

Organize Data

A matrix is a rectangular array of variables or constants in horizontal rows and vertical columns, usually enclosed in brackets. In a matrix, the numbers or data are organized so that each position in the matrix has a purpose. Each value in the matrix is called an element. A matrix is usually named using an uppercase letter.

\[
\begin{bmatrix}
8 & -2 & 5 & 6 \\
-1 & 3 & -3 & 6 \\
7 & -8 & 1 & 4
\end{bmatrix}
\]

3 rows
4 columns

A matrix can be described by its dimensions. A matrix with \(m\) rows and \(n\) columns is an \(m \times n\) matrix (read “\(m\) by \(n\)”). Matrix \(A\) above is a \(3 \times 4\) matrix because it has 3 rows and 4 columns.

Example 1 Dimensions and Elements of a Matrix

Use \(A = \begin{bmatrix}
-18 & 6 & 38 & 22 \\
9 & -9 & 22
\end{bmatrix}\) to answer the following.

a. State the dimensions of \(A\).

\[
\begin{bmatrix}
-18 & 6 & 38 & 22 \\
9 & -9 & 22
\end{bmatrix}
\]

2 rows
3 columns

Since \(A\) has 2 rows and 3 columns, the dimensions of \(A\) are \(2 \times 3\).

b. Find the value of \(a_{21}\).

The element \(-8\) is in Row 2, Column 2, depicted by \(a_{21}\).

The value of \(a_{21}\) is 9.

Guided Practice

Use \(B = \begin{bmatrix}
10 & -8 \\
-2 & 19 \\
6 & -1
\end{bmatrix}\) to answer the following.

1A. State the dimensions of \(B\).

1B. Find the value of \(b_{32}\).
Certain matrices have special names.

- **row matrix**: one row
- **column matrix**: one column
- **square matrix**: same number of rows and columns
- **zero matrix**: Every element is zero.

\[
\begin{bmatrix}
8 & -5 & 2 & 4 \\
-1 & 8 & -3 & 2 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Two matrices are considered **equal matrices** if they have the same dimensions and if each element of one matrix is equal to the corresponding element in the other matrix.

\[
\begin{bmatrix}
-3 & 4 \\
6 & 3 \\
5 & 2
\end{bmatrix}
\neq
\begin{bmatrix}
4 & 3 & 2 \\
-3 & 6 & 5
\end{bmatrix}
\neq
\begin{bmatrix}
1 & 4 \\
16 & -5
\end{bmatrix}
\neq
\begin{bmatrix}
5 & 6 & 0 \\
0 & 7 & 2 \\
3 & 1 & 4
\end{bmatrix} = \begin{bmatrix}
5 & 6 & 0 \\
0 & 7 & 2 \\
3 & 1 & 4
\end{bmatrix}
\]

Matrices with different dimensions **cannot** be equal.

The corresponding elements are not equal, so the matrices are not equal.

The matrices are equal.

Matrices are used to organize and analyze data.

### Real-World Example 2: Organize Data into a Matrix

**FOOTBALL** The West High School football team used five running backs throughout its season. Coach Williams wanted to compare the statistics of each player.

<table>
<thead>
<tr>
<th>Player</th>
<th>Games</th>
<th>Attempts</th>
<th>Yards</th>
<th>Average</th>
<th>TDs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Joey</td>
<td>11</td>
<td>72</td>
<td>439</td>
<td>6.10</td>
<td>8</td>
</tr>
<tr>
<td>DeShawn</td>
<td>9</td>
<td>143</td>
<td>1024</td>
<td>7.16</td>
<td>12</td>
</tr>
<tr>
<td>Dario</td>
<td>11</td>
<td>164</td>
<td>885</td>
<td>5.40</td>
<td>15</td>
</tr>
<tr>
<td>Leo</td>
<td>11</td>
<td>84</td>
<td>542</td>
<td>6.45</td>
<td>7</td>
</tr>
<tr>
<td>Alex</td>
<td>10</td>
<td>151</td>
<td>966</td>
<td>6.40</td>
<td>11</td>
</tr>
</tbody>
</table>

**a.** Organize the data in a matrix, listing players in the first column, in order from most attempts to least attempts.

**b.** What are the dimensions of the matrix? What value is $a_{34}$?

There are five rows and five columns, so the dimensions are $5 \times 5$. The value $a_{34}$, which is in the third row and fourth column, is 7.16.
Displaying Data

Matrices displaying real-world data can often be flipped, with the rows changing places with the columns.

Guided Practice

2. SUBS  The figure at the right shows the prices of small, medium, and large subs.
   A. Organize the data in a matrix, listing the subs from least to most expensive.
   B. What are the dimensions of the matrix?
   C. What is the value of $a_{21}$?

Analyze Data

Once data are organized in a matrix, they can be analyzed and interpreted. Sometimes, the sums or averages of rows or columns provide further analysis. Other times, the sums or averages provide data that are meaningless.

Example 3  Analyze Data with Matrices

FOOTBALL  Coach Williams would like to use the matrix from Example 2 to further analyze his players’ statistics.

a. Add the elements in columns 2 and 3 and interpret the results.
   The sum of column 2 is 614.
   This is the total number of attempts for the players.
   The sum of column 3 is 3856.
   This is the total number of yards gained.

b. Coach Williams wants to determine the average yards per attempt for his five running backs combined. He decides to add the elements in column 4 and divide by 5, the number of players. What is this average?
   The average is 6.302.

c. Is this an accurate average? Explain.
   No. The players did not have the same number of attempts, so finding the average of column 4 would not determine an accurate average. Instead, Coach Williams needs to divide the sum of column 3 by the sum of column 2. The accurate average is about 6.28.

d. Would adding the rows provide any meaningful data for Coach Williams? Explain your reasoning.
   No. The sum of a row includes five different forms of data.

Guided Practice

3. POPULATION  The table displays some of the U.S. Census data.
   A. Organize the data in a matrix.
   B. Add the elements in the columns and interpret the results.
   C. Add the elements in the rows and interpret the results.
   D. Would finding the average of the rows or columns provide any meaningful data?
Check Your Understanding

Example 1  State the dimensions of each matrix.

1. \[
\begin{bmatrix}
1 & 4 & -4 & 0 \\
-2 & 3 & 6 & -8
\end{bmatrix}
\]
2. \[
\begin{bmatrix}
1 \\
-2 \\
5
\end{bmatrix}
\]
3. \[
\begin{bmatrix}
-1 & 4 \\
2 & 9 \\
17 & 21
\end{bmatrix}
\]

Identify each element of matrix \( A = \begin{bmatrix}
1 & -6 & x & -4 \\
-2 & 3 & -1 & 9 \\
5 & -8 & 2 & 12
\end{bmatrix}\).

4. \( a_{32} \)
5. \( a_{11} \)
6. \( a_{33} \)
7. \( a_{24} \)

Examples 2–3  8. **FINANCIAL LITERACY**  Use the table that shows the city and highway gas mileage of five different types of vehicles.

<table>
<thead>
<tr>
<th>Vehicle</th>
<th>SUV</th>
<th>Mini-van</th>
<th>Sedan</th>
<th>Compact</th>
<th>APV</th>
</tr>
</thead>
<tbody>
<tr>
<td>City</td>
<td>23</td>
<td>21</td>
<td>21</td>
<td>42</td>
<td>61</td>
</tr>
<tr>
<td>Highway</td>
<td>25</td>
<td>24</td>
<td>32</td>
<td>49</td>
<td>70</td>
</tr>
</tbody>
</table>

Source: Auto Hoppers

a. Organize the gas mileages in a matrix.
b. Which type of vehicle has the best gas mileage?
c. Add the elements of each row and interpret the results.
d. Add the elements of each column and interpret the results.

Practice and Problem Solving

Example 1  State the dimensions of each matrix.

9. \( \begin{bmatrix}
-9 & 6
\end{bmatrix}\)
10. \( \begin{bmatrix}
15 & y \\
8 & -9
\end{bmatrix}\)
11. \( \begin{bmatrix}
6 & 11 & -4 & -2 \\
-8 & 5 & -1 & 0
\end{bmatrix}\)
12. \( \begin{bmatrix}
4 & -3 & 1 \\
x & 3y & 0 \\
8 & 12 & 11
\end{bmatrix}\)
13. \( \begin{bmatrix}
2 \\
x \\
-3
\end{bmatrix}\)
14. \( [115] \)

Identify each element for the following matrices

\[
A = \begin{bmatrix}
6 & y \\
-9 & 31 \\
11 & 5
\end{bmatrix},
B = \begin{bmatrix}
10 & -8 & 2x \\
-2 & 19 & 4
\end{bmatrix}
\]

15. \( a_{21} \)
16. \( b_{22} \)
17. \( b_{13} \)
18. \( a_{12} \)

Example 2  Organize the information in matrix.

<table>
<thead>
<tr>
<th>Name</th>
<th>Game 1</th>
<th>Game 2</th>
<th>Game 3</th>
<th>Series</th>
</tr>
</thead>
<tbody>
<tr>
<td>John</td>
<td>221</td>
<td>201</td>
<td>185</td>
<td>607</td>
</tr>
<tr>
<td>Hideo</td>
<td>168</td>
<td>233</td>
<td>159</td>
<td>560</td>
</tr>
<tr>
<td>Paulo</td>
<td>187</td>
<td>189</td>
<td>211</td>
<td>587</td>
</tr>
</tbody>
</table>
20. **CHART** | **Name** | **Cell Phone Minutes** | **Text Messages** | **Picture Messages** |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Chee</td>
<td>95</td>
<td>227</td>
<td>138</td>
</tr>
<tr>
<td>Emelia</td>
<td>83</td>
<td>213</td>
<td>189</td>
</tr>
<tr>
<td>Lina</td>
<td>101</td>
<td>199</td>
<td>202</td>
</tr>
</tbody>
</table>

21. **SHOES** A consumer service company rated several pairs of shoes by cost, level of comfort, look, and longevity using a scale of 1–5, with 1 being low and 5 being high.

<table>
<thead>
<tr>
<th>Brand</th>
<th>Cost</th>
<th>Comfort</th>
<th>Look</th>
<th>Longevity</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>B</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>C</td>
<td>5</td>
<td>5</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>D</td>
<td>1</td>
<td>5</td>
<td>5</td>
<td>2</td>
</tr>
</tbody>
</table>

a. Write a $4 \times 4$ matrix to organize this information.
b. Which shoe would you buy based on this information, and why?
c. Would finding the sum of the rows or columns provide any useful information? Explain your reasoning.

**Identify each element for the following matrices.**

\[
A = \begin{bmatrix}
23 & 11 \\
5 & -12 \\
-5 & 15 \\
\end{bmatrix},
B = \begin{bmatrix}
9 & -3 & 7 \\
4x & 18 & -6 \\
\end{bmatrix}
\]

22. $a_{32}$ 23. $b_{21}$ 24. $b_{12}$ 25. $a_{21}$

26. **WATER PARK** Use the sign at the entrance of the park shown at the right.

a. Write a matrix for the prices of admission for adults, children, and students.
b. What are the dimensions of the matrix?

27. **TRAVEL** Use the following flight costs for a flight to a certain city.

- Coach: $249 weekday; $259 weekend
- Business class: $279 weekday; $289 weekend
- First class: $319 weekday; $339 weekend

a. Write a $3 \times 2$ matrix that represents the cost of each flight.
b. Write a $2 \times 3$ matrix that represents the cost of each flight.

28. **INVENTORY** Mr. Kelley owns three golf supply stores. Store 1 has 200 white, 100 red, and 150 yellow golf balls. Store 2 has 300 white, 175 red, and 225 yellow golf balls. Store 3 has 275 white, 150 red, and 220 yellow golf balls.

a. Organize this information into a matrix with store numbers as the column heads.
b. Find the sum of the columns. What does the sum represent?
c. Find the sum of the rows. What does the sum represent?

**Identify each element for the following matrices.**

\[
A = \begin{bmatrix}
x^2 + 4 & y + 6 \\
x - y & 2 - y \\
\end{bmatrix},
B = \begin{bmatrix}
0 & x & -2y \\
5x & 3y & -4x \\
-y & 0 & 0 \\
\end{bmatrix}
\]

29. $a_{11}$ 30. $a_{22}$ 31. $b_{31}$ 32. $b_{23}$
33 PLANETS Use the table that shows the distance of the other planets from Earth and the Sun.
   a. Organize the distances in a matrix.
   b. What are the dimensions of the matrix?
   c. What is the value of $a_{42}$?

<table>
<thead>
<tr>
<th>Planet</th>
<th>Distance from Sun (millions of miles)</th>
<th>Distance from Earth (millions of miles)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mercury</td>
<td>36.00</td>
<td>57</td>
</tr>
<tr>
<td>Venus</td>
<td>67.24</td>
<td>26</td>
</tr>
<tr>
<td>Mars</td>
<td>141.71</td>
<td>35</td>
</tr>
<tr>
<td>Jupiter</td>
<td>483.88</td>
<td>370</td>
</tr>
<tr>
<td>Saturn</td>
<td>887.14</td>
<td>744</td>
</tr>
<tr>
<td>Uranus</td>
<td>1783.98</td>
<td>1607</td>
</tr>
<tr>
<td>Neptune</td>
<td>2796.46</td>
<td>2680</td>
</tr>
</tbody>
</table>

Source: FactMonster

34. MULTIPLE REPRESENTATIONS In this problem, you will explore reversing rows and columns of matrices.
   a. Tabular Convert the data into a matrix with the names of the players along the columns.
   b. Algebraic Find the sums of the columns.
   c. Tabular Switch the data in the matrix, now having the names of the players along the rows.
   d. Algebraic Find the sums of the rows.
   e. Analytical Make a conjecture about the effect on the data when the rows and columns of a matrix are switched.

<table>
<thead>
<tr>
<th>Name</th>
<th>Goals</th>
<th>Assists</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amy</td>
<td>8</td>
<td>3</td>
</tr>
<tr>
<td>Tama</td>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td>Kristen</td>
<td>1</td>
<td>8</td>
</tr>
<tr>
<td>Catalina</td>
<td>4</td>
<td>2</td>
</tr>
</tbody>
</table>

H.O.T. Problems Use Higher-Order Thinking Skills

REASONING Determine whether each statement is true or false. Explain.

35. C is a square matrix with 4 columns. It contains element $c_{33}$.

36. D is a square matrix with 3 rows. It contains element $c_{22}$.

37. ERROR ANALYSIS Kyla and Jay were asked to identify $b_{32}$ for $B = \begin{pmatrix} -6 & 7 \\ 0 & 5 \\ 8 & 2 \end{pmatrix}$.
   Is either or them correct? Explain your answer.

   Kyla
   There is no element $b_{32}$ for $B$ because $B$ is a $2 \times 3$ matrix

   Jay
   The value of $b_{32}$ is 5.

38. CHALLENGE Solve the following for $x$, $y$, and $z$.

   \[
   \begin{pmatrix}
   2x - y & 3x + 4z \\
   7x - 8z & 5y + 12z 
   \end{pmatrix} = \begin{pmatrix}
   9z - 5x + 1 & 5y - 2x \\
   3y - 4z & 12x + 2y 
   \end{pmatrix}
   \]

39. OPEN ENDED Form a matrix using real-world data in which the sum of the columns is relevant and the sum of the rows is irrelevant.

40. WRITING IN MATH Explain how a matrix can be helpful when deciding what college you want to attend.
41. What is the equation of the line that has a slope of 3 and passes through the point (2, -9)?

- A \( y = 3x + 11 \)
- B \( y = 3x + 15 \)
- C \( y = 3x - 11 \)
- D \( y = 3x - 15 \)

42. GEOMETRY Line \( q \) is shown below. Which equation best represents a line parallel to line \( q \)?

- F \( y = x + 2 \)
- G \( y = 2x + 4 \)
- H \( y = 2x - 3 \)
- J \( y = -2x + 2 \)

43. SHORT RESPONSE What is the area of the shaded part of the rectangle below?

44. SAT/ACT The results of a recent poll are organized in the matrix.

\[
\begin{array}{c|cc}
\text{Proposition 1} & \text{For} & \text{Against} \\
\hline
\text{Proposition 2} & 1553 & 771 \\
\text{Proposition 3} & 2088 & 229 \\
\end{array}
\]

Based on these results, which conclusion is not valid?

- A There were 771 votes cast against Proposition 1.
- B More people voted against Proposition 1 than voted for Proposition 2.
- C Proposition 2 has little chance of passing.
- D How people voted against Proposition 2 than for Proposition 1.
- E More people voted for Proposition 1 than for Proposition 3.

45. COLLEGE FOOTBALL In a recent year, Darren McFadden of Arkansas placed second overall in the Heisman Trophy voting. Players are given 3 points for every first-place vote, 2 points for every second-place vote, and 1 point for every third-place vote. McFadden received 490 total votes for first, second, and third place, for a total of 878 points. If he had 4 more than twice as many second-place votes as third-place votes, how many votes did he receive for each place? (Lesson 3-5)

46. PACKAGING The Cookie Factory sells chocolate chip and peanut butter cookies in combination packages that contain between six and twelve cookies. At least three of each type of cookie should be in each package. How many of each type of cookie should be in each package to maximize the profit? (Lesson 3-4)

47. Find the slope of the line that passes through each pair of points. (Lesson 2-3)

- (\(-3, -6\), \((-1, -9)\))
- (\((-2, 6)\), \((4, -1)\))
- (\((5, -3)\), \((8, 2)\))

48. Multiply. (Lesson 0-2)

- \((2x + 1)(-3x - 2)\)
- \((y + 6)(y - 8)\)
- \((x + y)(x - 2y)\)

49. Evaluate each expression if \(w = 3\), \(x = -2\), \(y = 4\), and \(z = 0.5\). (Lesson 1-2)

- \(4x - 6y + 2z\)
- \(5w + 2(x - z) + 2y\)
- \(4[3(2z + y) - 2(w + x)]\)
People in the workforce often use computer **spreadsheets** to organize, display, and analyze data. Similar to a matrix, data in a spreadsheet are entered into rows and columns. Then the data can be used to create graphs or perform calculations.

### Example

The manager of a gourmet food store has gathered data on the number of pounds of bulk coffees they have sold each week in January. Enter the data into a spreadsheet.

<table>
<thead>
<tr>
<th>Coffee</th>
<th>1/5</th>
<th>1/12</th>
<th>1/19</th>
<th>1/26</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hawaiian Kona</td>
<td>17</td>
<td>22</td>
<td>11</td>
<td>23</td>
</tr>
<tr>
<td>Mocha Java</td>
<td>31</td>
<td>34</td>
<td>22</td>
<td>29</td>
</tr>
<tr>
<td>House Blend</td>
<td>55</td>
<td>61</td>
<td>44</td>
<td>71</td>
</tr>
<tr>
<td>Espresso</td>
<td>41</td>
<td>36</td>
<td>60</td>
<td>77</td>
</tr>
<tr>
<td>Decaf Espresso</td>
<td>23</td>
<td>29</td>
<td>19</td>
<td>44</td>
</tr>
<tr>
<td>Breakfast Blend</td>
<td>8</td>
<td>18</td>
<td>19</td>
<td>31</td>
</tr>
<tr>
<td>Decaf Breakfast Blend</td>
<td>22</td>
<td>18</td>
<td>30</td>
<td>32</td>
</tr>
<tr>
<td>Organic Italian Roast</td>
<td>26</td>
<td>16</td>
<td>31</td>
<td>39</td>
</tr>
</tbody>
</table>

Use Column A for the types of coffee, Column B for the sales in the week starting 1/5, Column C for sales in the week starting 1/12, and Columns D and E for the sales in the weeks starting 1/19 and 1/26.

### Exercises

1. Enter the data on smartphones on page 185 into a spreadsheet.
2. Compare and contrast how data are organized in a spreadsheet and in a matrix.
3. A SUM formula allows you to find the sum of the entries in a column or row.
   a. The formula `=SUM(B1:B8)` finds the sum of column B. Enter formulas in cells B9, C9, D9, and E9 to find the sums of those columns. What do the sums of the columns represent in the situation?
   b. Enter formulas in cells F1 through F8 to find the sums of rows 1 through 8. What do the sums of the rows represent in the situation?
   c. Find the sum of row 9 and the sum of column F. What do you observe? Explain.
To add or subtract two matrices with the same dimensions, add or subtract their corresponding elements.

1. **Add and Subtract Matrices** Matrices can be added or subtracted if and only if they have the same dimensions.

   **Key Concept** Adding and Subtracting Matrices
   
   Words: To add or subtract two matrices with the same dimensions, add or subtract their corresponding elements.
   
   $$A + B = A + B$$
   
   Symbols:
   
   $$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$
   
   $$A + B = \begin{bmatrix} a + e & a + e \\ c + g & c + h \end{bmatrix}, \quad A - B = \begin{bmatrix} a - e & b - f \\ c - g & d - h \end{bmatrix}$$
   
   Example:
   
   $$\begin{bmatrix} 3 & -5 \\ 1 & 7 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ -9 & 10 \end{bmatrix} = \begin{bmatrix} 3 + 2 & -5 + 0 \\ 1 + (-9) & 7 + 10 \end{bmatrix}$$

2. **Example 1** Add and Subtract Matrices

   Find each of the following for $A = \begin{bmatrix} 16 \\ -9 \end{bmatrix}$, $B = \begin{bmatrix} -4 & -1 \\ -3 & -7 \end{bmatrix}$, and $C = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$.

   a. $A + B$
   
   $$A + B = \begin{bmatrix} 16 & 2 \\ -9 & 8 \end{bmatrix} + \begin{bmatrix} -4 & -1 \\ -3 & -7 \end{bmatrix}$$
   
   $$= \begin{bmatrix} 16 + (-4) & 2 + (-1) \\ -9 + (-3) & 8 + (-7) \end{bmatrix}$$
   
   $$= \begin{bmatrix} 12 & 1 \\ -12 & 1 \end{bmatrix}$$

   b. $B - C$
   
   $$B - C = \begin{bmatrix} -4 & -1 \\ -3 & -7 \end{bmatrix} - \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

   Since the dimensions of $B$ and $C$ are different, you cannot subtract the matrices.

   **Guided Practice**

   1A. $\begin{bmatrix} -3 & 4 \\ -9 & -5 \end{bmatrix} + \begin{bmatrix} -4 & 12 \\ 8 & -7 \end{bmatrix}$

   1B. $\begin{bmatrix} -9 & 8 & 3 \\ -2 & 4 & -7 \end{bmatrix} + \begin{bmatrix} -4 & -3 & 6 \\ -9 & -5 & 18 \end{bmatrix}$
2 Scalar Multiplication You can multiply any matrix by a constant called a scalar. When you do this, you multiply each individual element by the value of the scalar. This operation is called scalar multiplication.

**Key Concept Multiplying by a Scalar**

| Words | To multiply a matrix by a scalar \( k \), multiply each element by \( k \).
|-------|--------------------------------------------------|
| Symbols | \( k \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix} \)
| Example | \(-3 \begin{bmatrix} 4 & 1 \\ 7 & -2 \end{bmatrix} = \begin{bmatrix} -3(4) & -3(1) \\ -3(7) & -3(-2) \end{bmatrix} \)

**Example 2 Multiply a Matrix by a Scalar**

If \( R = \begin{bmatrix} -12 & 8 & 6 \\ -16 & 4 & 19 \end{bmatrix} \), find \( 5R \).

\[
5R = 5 \begin{bmatrix} -12 & 8 & 6 \\ -16 & 4 & 19 \end{bmatrix} = \begin{bmatrix} 5(-12) & 5(8) & 5(6) \\ 5(-16) & 5(4) & 5(19) \end{bmatrix} = \begin{bmatrix} -60 & 40 & 30 \\ -80 & 20 & 95 \end{bmatrix}
\]

**Guided Practice**

2. If \( T = \begin{bmatrix} 8 & 0 & 3 \\ -1 & -2 & -2 \\ -1 & 1 & 9 \end{bmatrix} \), find \( -4T \).

Many properties of real numbers also hold true for matrices. A summary of these properties is listed below.

**Key Concept Properties of Matrix Operations**

For any matrices \( A, B, \) and \( C \) for which the matrix sum and product are defined and any scalar \( k \), the following properties are true.

- Commutative Property of Addition \( A + B = B + A \)
- Associative Property of Addition \( (A + B) + C = A + (B + C) \)
- Left Scalar Distributive Property \( k(A + B) = kA + kB \)
- Right Scalar Distributive Property \( (A + B)k = kA + kB \)

Multi-step operations can be performed on matrices. The order of these operations is the same as with real numbers.
Example 3 Multi-Step Operations

If \( A = \begin{bmatrix} -9 \\ 2 \\ -6 \end{bmatrix} \) and \( B = \begin{bmatrix} -4 \\ 2 \\ -3 \end{bmatrix} \), find \(-4B - 3A\).

\[ -4B - 3A = -4 \begin{bmatrix} -4 \\ 2 \\ -3 \end{bmatrix} - 3 \begin{bmatrix} -9 \\ 2 \\ -6 \end{bmatrix} \]

Substitution

\[ = \begin{bmatrix} -4(-4) \\ 2 \\ -3 \end{bmatrix} - \begin{bmatrix} 3(-9) \\ 2 \\ -6 \end{bmatrix} \]

Distribute the scalars in each matrix.

\[ = \begin{bmatrix} 16 \\ 32 \\ -8 \\ 12 \\ -18 \end{bmatrix} - \begin{bmatrix} -27 \\ 6 \\ 36 \\ 6 \end{bmatrix} \]

Multiply.

\[ = \begin{bmatrix} 16 - (-27) \\ 32 - 36 \\ -8 - 6 \\ 12 - 18 \end{bmatrix} \]

Subtract corresponding elements.

\[ = \begin{bmatrix} 43 \\ -4 \end{bmatrix} \]

Simplify.

Guided Practice

3. If \( A = \begin{bmatrix} -5 \\ 6 \\ -8 \end{bmatrix} \) and \( B = \begin{bmatrix} 12 \\ 5 \\ -4 \end{bmatrix} \), find \(-6B + 7A\).

Matrices can be used in many business applications.

Example 4 Use Multi-Step Operations with Matrices

**BUSINESS** Refer to the application at the beginning of the lesson. Express the average wages and sales for the entire company for a 5-day week.

To calculate the 5-day sales for the entire company, each matrix needs to be multiplied by 5 and the totals added together.

\[ \begin{bmatrix} 900 & 145,000 \\ 2400 & 225,000 \\ 2700 & 290,000 \end{bmatrix} + 5 \begin{bmatrix} 1800 & 122,000 \\ 1800 & 145,500 \\ 1800 & 160,000 \end{bmatrix} + 5 \begin{bmatrix} 1050 & 109,500 \\ 1800 & 135,000 \\ 1800 & 150,500 \end{bmatrix} \]

Write matrices.

\[ = \begin{bmatrix} 4500 & 725,000 \\ 12,000 & 1,125,000 \\ 13,500 & 1,450,000 \end{bmatrix} + \begin{bmatrix} 4500 & 610,000 \\ 9000 & 727,500 \\ 9000 & 800,000 \end{bmatrix} + \begin{bmatrix} 5250 & 547,500 \\ 9000 & 675,000 \\ 9000 & 752,500 \end{bmatrix} \]

Multiply scalars.

\[ = \begin{bmatrix} Wages \\ Sales \end{bmatrix} \]

Entry [14,250, 1,882,500]

Assistant [30,000, 2,527,500]

Associate [31,500, 3,002,500]

Add matrices.

The final matrix indicates the average weekly sales and wages for all of the representatives of the company.

Guided Practice

4. Use the data above to calculate the average yearly sales and wages for the company, assuming 260 working days.
Check Your Understanding

Example 1 Perform the indicated operations. If the matrix does not exist, write impossible.

1. \[
\begin{bmatrix}
-8 & 2 \\
11 & -7
\end{bmatrix} + \begin{bmatrix}
11 & -7 \\
1 & 1
\end{bmatrix}
\]

2. \[
\begin{bmatrix}
9 & -8 & 4 \\
12 & 2
\end{bmatrix} + \begin{bmatrix}
1 & 2 \\
2 & 1
\end{bmatrix}
\]

3. \[
\begin{bmatrix}
7 & -12 \\
15 & 4
\end{bmatrix} - \begin{bmatrix}
9 & 6 \\
4 & -9
\end{bmatrix}
\]

4. \[
\begin{bmatrix}
5 & 13 & -6 \\
3 & -17 & 2
\end{bmatrix} - \begin{bmatrix}
-2 & 18 & 8 \\
2 & -11 & 0
\end{bmatrix}
\]

Example 2 Perform the indicated operations. If the matrix does not exist, write impossible.

5. \[
\begin{bmatrix}
6 & 4 & 0 \\
-2 & 14 & -8 \\
-4 & -6 & 7
\end{bmatrix}
\]

6. \[
\begin{bmatrix}
15 & -9 & 2 & 3 \\
6 & -11 & 14 & -2 \\
4 & -8 & -10 & 27
\end{bmatrix}
\]

Example 3 Use matrices \(A, B, C,\) and \(D\) to find the following.

\[
A = \begin{bmatrix}
6 & -4 \\
3 & -5
\end{bmatrix} \quad B = \begin{bmatrix}
8 & -1 \\
-2 & 7
\end{bmatrix} \quad C = \begin{bmatrix}
-4 & -6 \\
12 & -7
\end{bmatrix} \quad D = \begin{bmatrix}
9 & 6 & 0 \\
-2 & 8 & 0
\end{bmatrix}
\]

7. \(4B - 2A\)

8. \(-8C + 3A\)

9. \(-5B - 2D\)

10. \(-4C - 5B\)

Example 4 GRADES Geraldo, Olivia, and Nikki have had two tests in their math class. The table shows the test grades for each student.

<table>
<thead>
<tr>
<th>Student</th>
<th>Test 1</th>
<th>Test 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Geraldo</td>
<td>85</td>
<td>72</td>
</tr>
<tr>
<td>Olivia</td>
<td>75</td>
<td>74</td>
</tr>
<tr>
<td>Nikki</td>
<td>96</td>
<td>83</td>
</tr>
</tbody>
</table>

a. Write a matrix for the information.

b. Find the sum of the scores from the two tests expressed as a matrix.

c. Express the difference in scores from test 1 to test 2 as a matrix.

Practice and Problem Solving

Example 1 Perform the indicated operations. If the matrix does not exist, write impossible.

12. \[
\begin{bmatrix}
12 & -5 \\
-8 & -3
\end{bmatrix} + \begin{bmatrix}
-6 & 11 \\
-7 & 2
\end{bmatrix}
\]

13. \[
\begin{bmatrix}
9 & 5 \\
-2 & 16
\end{bmatrix} + \begin{bmatrix}
-6 & -3 \\
12 & 2
\end{bmatrix}
\]

Examples 2–3 BUSINESS The drink menu from a fast-food restaurant is shown at the right. The store owner has decided that all of the prices must be increased by 10%.

a. Write matrix \(C\) to represent the current prices.

b. What scalar can be used to determine a matrix \(N\) to represent the new prices?

c. Find \(N\).

d. What is \(N - C\)? What does this represent in this situation?

<table>
<thead>
<tr>
<th>Drink</th>
<th>Small</th>
<th>Medium</th>
<th>Large</th>
</tr>
</thead>
<tbody>
<tr>
<td>Soda</td>
<td>$0.95</td>
<td>$1.00</td>
<td>$1.05</td>
</tr>
<tr>
<td>Iced tea</td>
<td>$0.75</td>
<td>$0.80</td>
<td>$0.85</td>
</tr>
<tr>
<td>Lemonade</td>
<td>$0.75</td>
<td>$0.80</td>
<td>$0.85</td>
</tr>
<tr>
<td>Coffee</td>
<td>$1.00</td>
<td>$1.10</td>
<td>$1.20</td>
</tr>
</tbody>
</table>
Example 4

Perform the indicated operations. If the matrix does not exist, write impossible.

15. \[
\begin{bmatrix}
-5 \\ 8 \\ 1 \\ 0
\end{bmatrix}
\begin{bmatrix}
19 \\ -2 \\ 4 \\ 7
\end{bmatrix}
\]
16. \[
\begin{bmatrix}
4 & -3 & 3 \\ -8 & 12 & 1 \\ 0 & -1 & 5 \\ 7 & -9 & 4
\end{bmatrix}
\begin{bmatrix}
-3 & -8 & 12 \\ -1 & -5 & 3 \\ -1 & 22 & -9 \\ -6 & 31 & 9
\end{bmatrix}
\]
17. \[
\begin{bmatrix}
62 \\ -37 \\ -4
\end{bmatrix}
+ \begin{bmatrix}
34 \\ 76 \\ -13
\end{bmatrix}
\]
18. \[
\begin{bmatrix}
2 & 4 & 11 \\ -6 & 12 & -3
\end{bmatrix}
- \begin{bmatrix}
8 & -9 & -3 \\ 5 & 14 & 0
\end{bmatrix}
\]
19. \[
\begin{bmatrix}
5 \\ -9
\end{bmatrix}
+ \begin{bmatrix}
-3 \\ -7
\end{bmatrix}
- \begin{bmatrix}
9 \\ 16
\end{bmatrix}
\]
20. \[
\begin{bmatrix}
5 \\ -4 \\ 3
\end{bmatrix}
+ \begin{bmatrix}
-2 \\ 3
\end{bmatrix}
\]

21. **BOOKS**
Library A has 10,000 novels, 5000 biographies, and 5000 children’s books.
Library B has 15,000 novels, 10,000 biographies, and 2500 children’s books. Library C has 4000 novels, 700 biographies, and 800 children’s books.

a. Express each library’s number of books as a matrix. Label the matrices A, B, and C.
b. Find the total number of each type of book in all 3 libraries. Express as a matrix.
c. How many more books of each type does Library A have than Library C?
d. Find A + B. Does the matrix have meaning in this situation? Explain.

Perform the indicated operations. If the matrix does not exist, write impossible.

22. \[
-3 \begin{bmatrix}
18 & -6 & -8 \\ -5 & -3 & 12 \\ 0 & 3x & -y
\end{bmatrix}
\]

23. \[
8 \begin{bmatrix}
-a & 4b & c-b \\ -13 & 10 & -5c
\end{bmatrix}
\]

24. \[
-4 \begin{bmatrix}
-7 \\ 4 \\ -9
\end{bmatrix}
+ \begin{bmatrix}
-8 \\ 3x \\ -9
\end{bmatrix}
- 5 \begin{bmatrix}
x & 4 \\ 6 & 12
\end{bmatrix}
\]

25. \[
-5 \begin{bmatrix}
4 & -8 \\ 8 & -9
\end{bmatrix}
+ \begin{bmatrix}
4 & -2 \\ -3 & -6
\end{bmatrix}
\]

26. \[
-6 \begin{bmatrix}
6 & 3y \\ 4y+1 & -2 \\ -9 & xy
\end{bmatrix}
+ \begin{bmatrix}
-5 & -6 \\ 8 & -7
\end{bmatrix}
+ \begin{bmatrix}
x + 2 & 2x \\ 2 & 2
\end{bmatrix}
\]

27. \[
-4 \begin{bmatrix}
9 & -5y \\ 11 & -3 \\ -1 & 2-x
\end{bmatrix}
- 7 \begin{bmatrix}
8 & -y & 12 \\ 3x & 2 + x & -y
\end{bmatrix}
\]

28. **WEATHER**
The table shows snowfall in inches.

a. Express the normal snowfall data and the 2007 data in two 4 × 3 matrices.
b. Subtract the matrix of normal data from the matrix of 2007 data. What does the difference represent in the context of the situation?
c. Explain the meaning of positive and negative numbers in the difference matrix. What trends do you see in the data?

Perform the indicated operations. If the matrix does not exist, write impossible.

29. \[
\begin{bmatrix}
12.5 & -16.4 \\ 431 & -2.43 \\ -6.8 & -14.1
\end{bmatrix}
- 3 \begin{bmatrix}
-18.7 & -11.8 \\ 8.1 & -6.91 \\ -6.21 & -17.6
\end{bmatrix}
\]

30. \[
-2 \begin{bmatrix}
-9.2 & -8.4 \\ 5.6 & -4.3 \\ 7.2 & -8.2
\end{bmatrix}
- 4 \begin{bmatrix}
4.1 & -2.9 \\ 5.6 & -4.3 \\ 7.2 & -8.2
\end{bmatrix}
\]

31. \[
\frac{3}{4} \begin{bmatrix}
12 & -16 \\ 15 & 8
\end{bmatrix}
+ \frac{2}{3} \begin{bmatrix}
21 & 18 \\ -4 & -6
\end{bmatrix}
\]

32. \[
-4 \begin{bmatrix}
\frac{4}{5} & 2 & -\frac{3}{4} \\ -1 & -\frac{2}{5} & -6
\end{bmatrix}
+ \frac{1}{5} \begin{bmatrix}
-4 & 1 & -8 \\ 4 & 2 & 3
\end{bmatrix}
\]
The table shows some of the world, Olympic, and American women’s freestyle swimming records.

<table>
<thead>
<tr>
<th>Distance (m)</th>
<th>World</th>
<th>Olympic</th>
<th>American</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>24.13 s</td>
<td>24.13 s</td>
<td>24.63 s</td>
</tr>
<tr>
<td>100</td>
<td>53.52 s</td>
<td>53.52 s</td>
<td>53.99 s</td>
</tr>
<tr>
<td>200</td>
<td>1:56.54 min</td>
<td>1:57.65 min</td>
<td>1:57.41 min</td>
</tr>
<tr>
<td>300</td>
<td>8:16.22 min</td>
<td>8:19.67 min</td>
<td>8:16.22 min</td>
</tr>
</tbody>
</table>

Source: USA Swimming

a. Find the difference between the American and World records expressed as a column matrix.

b. What is the meaning of each row in the column?

c. In which events were the fastest times set at the Olympics?

In this problem, you will investigate using matrices to represent transformations.

a. Algebraic The matrix \[
\begin{bmatrix}
-3 & -4 & 1 \\
8  & 6  & 0
\end{bmatrix}
\]
represents a triangle with vertices at \((-3, 8), (-4, 6),\) and \((1, 0)\). Write a matrix to represent \(\triangle ABC\).

b. Geometric Multiply the vertex matrix you wrote by 2. Then graph the figure represented by the new matrix.

c. Analytical How do the figures compare? Make a conjecture about the result of multiplying the matrix by 0.5. Verify your conjecture.

**H.O.T. Problems** Use Higher-Order Thinking Skills

35. PROOF Prove that matrix addition is commutative for \(2 \times 2\) matrices.

36. PROOF Prove that matrix addition is associative for \(2 \times 2\) matrices.

37. CHALLENGE Find the elements of \(C\) if:

\[
A = \begin{bmatrix} -3 & -4 \\ 8  & 6 \end{bmatrix}, B = \begin{bmatrix} 5 & -1 \\ 2  & -4 \end{bmatrix}, \text{ and } 3A - 4B + 6C = \begin{bmatrix} 13 & 22 \\ 10 & 4 \end{bmatrix}.
\]

38. REASONING Determine whether each statement is sometimes, always, or never true for matrices \(A\) and \(B\). Explain your reasoning.

a. If \(A + B\) exists, then \(A - B\) exists.

b. If \(k\) is a real number, then \(kA\) and \(kB\) exist.

c. If \(A - B\) does not exist, then \(B - A\) does not exist.

d. If \(A\) and \(B\) have the same number of elements, then \(A + B\) exists.

e. If \(kA\) exists and \(kB\) exists, then \(kA + kB\) exists.

39. OPEN ENDED Give an example of matrices \(A\) and \(B\) if \(4B - 3A = \begin{bmatrix} -6 & 5 \\ -2 & -1 \end{bmatrix}\).

40. WRITING IN MATH Explain how to find \(4D - 3C\) for two given matrices, \(C\) and \(D\) with the same dimensions.
Standardized Test Practice

41. What is the solution of the system of equations?
   \[
   \begin{align*}
   0.06p + 4q &= 0.88 \\
   p - q &= -2.25
   \end{align*}
   \]
   A \((-0.912, -1.338)\) \hspace{1cm} C \((-2, 0.25)\)
   B \((0.912, -3.162)\) \hspace{1cm} D \((-2, -4.25)\)

42. SHORT RESPONSE Find \(A + B\) if \(A = \begin{bmatrix} -7 & 3 \\ 2 & 6 \end{bmatrix}\) and \(B = \begin{bmatrix} 4 & 2 \\ 0 & 1 \end{bmatrix}\).

43. SAT/ACT Solve for \(x\) and \(y\).
   \[
   \begin{align*}
   x + 3y &= 16 \\
   7 - x &= 12
   \end{align*}
   \]
   \[
   \begin{align*}
   F \ x &= -5, y &= 7 \\
   G \ x &= 7, y &= 3 \\
   H \ x &= 7, y &= 5 \\
   \end{align*}
   \]

44. PROBABILITY A local pizzeria offers 5 different meat toppings and 6 different vegetable toppings. You decide to get two vegetable toppings and one meat topping. How many different types of pizzas can you order?
   A \(60\) \hspace{1cm} C \(120\)
   B \(75\) \hspace{1cm} D \(150\)

Spiral Review

Identify each element for the following matrices. (Lesson 4-1)

\[A = \begin{bmatrix} -3 & 6 \\ -5 & x \\ 8 & 4y \end{bmatrix}, \quad B = \begin{bmatrix} 16 & 4 & x \\ -2 & 9 & y \end{bmatrix}, \quad C = \begin{bmatrix} 9 & -5 & 3 & 2 \\ 0 & -6 & 8 & 1 \end{bmatrix}\]

45. \(a_{32}\) \hspace{1cm} 46. \(c_{13}\) \hspace{1cm} 47. \(b_{32}\)

Solve each system of equations. (Lesson 3-5)

48. \[
\begin{align*}
2x + 3y - z &= -1 \\
5x + y + 4z &= 30 \\
-8x - 2y + 5z &= -2
\end{align*}
\]
49. \[
\begin{align*}
3x - 4y + 6z &= 26 \\
5x + 3y + 2z &= 5 \\
-2x + 5y - 3z &= -9
\end{align*}
\]
50. \[
\begin{align*}
5x + 2y - 4z &= 22 \\
6x + 3y + 5z &= 5 \\
-2x - 4y + z &= 2
\end{align*}
\]

Solve each system of inequalities by graphing. (Lesson 3-3)

51. \[
\begin{align*}
x - 2y &> -4 \\
y &< 2x - 3
\end{align*}
\]
52. \[
\begin{align*}
y &\geq -4x + 6 \\
3y &< 2x + 9
\end{align*}
\]
53. \[
\begin{align*}
4x + 2y &> 8 \\
4y - 3x &\leq 12
\end{align*}
\]

54. RAKING LEAVES A student can earn $20 plus an extra $5 for each trash bag he or she completely fills with leaves. Write and solve an equation to determine how many bags the student will need to fill in order to earn $100. (Lesson 2-4)

55. SPORTS There are 15,991 more student athletes in New York than in Illinois. Write and solve an equation to find the number of student athletes in Illinois. (Lesson 1-3)

Skills Review

Simplify each expression. (Lesson 1-2)

56. \[4(2x - 3y) + 2(5x - 6y)\] \hspace{1cm} 57. \[-3(2a - 5b) - 4(4b + a)\] \hspace{1cm} 58. \[-7(x - y) + 5(y - x)\]
Multiplying Matrices

Why?

The table shows the scoring summary for Lisa Leslie, the WNBA’s all-time scoring leader, during her highest scoring seasons. Her total baskets can be summarized in the baskets matrix \( B \). The point values for each type of basket made can be organized in the point value matrix \( P \).

You can use matrix multiplication to find the points scored during each season.

<table>
<thead>
<tr>
<th>Type</th>
<th>2005</th>
<th>2006</th>
<th>2008</th>
<th>2009</th>
</tr>
</thead>
<tbody>
<tr>
<td>Field Goal</td>
<td>197</td>
<td>249</td>
<td>184</td>
<td>143</td>
</tr>
<tr>
<td>3-Point Field Goal</td>
<td>7</td>
<td>8</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>Free Throw</td>
<td>102</td>
<td>158</td>
<td>117</td>
<td>65</td>
</tr>
</tbody>
</table>

Source: WNBA

You can use matrix multiplication to find the points scored during each season.

\[
B = \begin{bmatrix}
197 & 249 & 184 & 143 \\
7 & 8 & 4 & 1 \\
102 & 158 & 117 & 65
\end{bmatrix} \quad P = \begin{bmatrix} 2 & 3 & 1 \end{bmatrix}
\]

### Example 1 Dimensions of Matrix Products

Determine whether each matrix product is defined. If so, state the dimensions of the product.

**a.** \( A_{3\times 4} \) and \( B_{4\times 2} \)

\[
A \cdot B = AB
\]

The inner dimensions are equal, so the product is defined. Its dimensions are \( 3 \times 2 \).

**b.** \( A_{5\times 3} \) and \( B_{5\times 4} \)

The inner dimensions are not equal, so the matrix product is not defined.

### Guided Practice

1A. \( A_{4\times 6} \) and \( B_{6\times 2} \) 

1B. \( A_{3\times 2} \) and \( B_{3\times 2} \)
The product of two matrices is found by multiplying columns and rows.

**Key Concept: Multiplying Matrices**

Words

The element in the \( m \)th row and \( r \)th column of matrix \( AB \) is the sum of the products of the corresponding elements in row \( m \) of matrix \( A \) and column \( r \) of matrix \( B \).

Symbols

\[
\begin{bmatrix}
    a & b \\
    c & d
\end{bmatrix} \cdot \begin{bmatrix}
    e & f \\
    g & h
\end{bmatrix} = \begin{bmatrix}
    ae + bg & af + bh \\
    ce + dg & cf + dh
\end{bmatrix}
\]

**Example 2: Multiply Square Matrices**

Find \( XY \) if \( X = \begin{bmatrix}
    6 & -3 \\
    -10 & -2
\end{bmatrix} \) and \( Y = \begin{bmatrix}
    -5 & -4 \\
    3 & 3
\end{bmatrix} \).

\( XY = \begin{bmatrix}
    6 & -3 \\
    -10 & -2
\end{bmatrix} \cdot \begin{bmatrix}
    -5 & -4 \\
    3 & 3
\end{bmatrix} \)

**Step 1**

Multiply the numbers in the first row of \( X \) by the numbers in the first column of \( Y \), add the products, and put the result in the first row, first column.

\[
\begin{bmatrix}
    6 & -3 \\
    -10 & -2
\end{bmatrix} \cdot \begin{bmatrix}
    -5 & -4 \\
    3 & 3
\end{bmatrix} = \begin{bmatrix}
    6(-5) + (-3)(3) \\
    -10(-5) + (-2)(3)
\end{bmatrix}
\]

**Step 2**

Follow the same procedure as in Step 1 using the first row and the second column numbers. Write the result in the first row, second column.

\[
\begin{bmatrix}
    6 & -3 \\
    -10 & -2
\end{bmatrix} \cdot \begin{bmatrix}
    -5 & -4 \\
    3 & 3
\end{bmatrix} = \begin{bmatrix}
    6(-5) + (-3)(3) & 6(-4) + (-3)(3) \\
    -10(-5) + (-2)(3)
\end{bmatrix}
\]

**Step 3**

Follow the same procedure with the second row and the first column numbers. Write the result in the second row, first column.

\[
\begin{bmatrix}
    6 & -3 \\
    -10 & -2
\end{bmatrix} \cdot \begin{bmatrix}
    -5 & -4 \\
    3 & 3
\end{bmatrix} = \begin{bmatrix}
    6(-5) + (-3)(3) & 6(-4) + (-3)(3) \\
    -10(-5) + (-2)(3)
\end{bmatrix}
\]

**Step 4**

The procedure is the same for the numbers in the second row, second column.

\[
\begin{bmatrix}
    6 & -3 \\
    -10 & -2
\end{bmatrix} \cdot \begin{bmatrix}
    -5 & -4 \\
    3 & 3
\end{bmatrix} = \begin{bmatrix}
    6(-5) + (-3)(3) & 6(-4) + (-3)(3) \\
    -10(-5) + (-2)(3)
\end{bmatrix}
\]

**Step 5**

Simplify the product matrix.

\[
\begin{bmatrix}
    6(-5) + (-3)(3) & 6(-4) + (-3)(3) \\
    -10(-5) + (-2)(3)
\end{bmatrix} = \begin{bmatrix}
    -39 & -33 \\
    44 & 34
\end{bmatrix}
\]

**Guided Practice**

2. Find \( UV \) if \( U = \begin{bmatrix}
    5 & 9 \\
    -3 & -2
\end{bmatrix} \) and \( V = \begin{bmatrix}
    2 & -1 \\
    6 & -5
\end{bmatrix} \).
Matrix multiplication can be used in many real-world situations.

### Real-World Example 3  Multiply Matrices

**SWIM MEET** At a particular swim meet, 7 points were awarded for each first-place finish, 4 points for second, and 2 points for third. Find the total number of points for each school. Which school won the meet?

<table>
<thead>
<tr>
<th>School</th>
<th>First Place</th>
<th>Second Place</th>
<th>Third Place</th>
</tr>
</thead>
<tbody>
<tr>
<td>Central</td>
<td>4</td>
<td>7</td>
<td>3</td>
</tr>
<tr>
<td>Franklin</td>
<td>8</td>
<td>9</td>
<td>1</td>
</tr>
<tr>
<td>Hayes</td>
<td>10</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>Lincoln</td>
<td>3</td>
<td>3</td>
<td>6</td>
</tr>
</tbody>
</table>

**Understand** The final scores can be found by multiplying the swim results for each school by the points awarded for each first-, second-, and third-place finish.

**Plan** Write the results of the races and the points awarded in matrix form. Set up the matrices so that the number of rows in the points matrix equals the number of columns in the results matrix.

\[
\begin{bmatrix}
4 & 7 & 3 \\
8 & 9 & 1 \\
10 & 5 & 3 \\
3 & 3 & 6
\end{bmatrix}
\quad \text{Points}
\begin{bmatrix}
7 \\
4 \\
2
\end{bmatrix}
\]

**Solve** Multiply the matrices.

\[
RP = \begin{bmatrix}
4 & 7 & 3 \\
8 & 9 & 1 \\
10 & 5 & 3 \\
3 & 3 & 6
\end{bmatrix}
\begin{bmatrix}
7 \\
4 \\
2
\end{bmatrix}
\]

\[
= \begin{bmatrix}
4(7) + 7(4) + 3(2) \\
8(7) + 9(4) + 1(2) \\
10(7) + 5(4) + 3(2) \\
3(7) + 3(4) + 6(2)
\end{bmatrix}
\]

\[
= \begin{bmatrix}
62 \\
94 \\
96 \\
45
\end{bmatrix}
\]

The product matrix shows the scores for Central, Franklin, Hayes, and Lincoln, respectively. Hayes won the swim meet with a total of 96 points.

**Check** \(R\) is a \(4 \times 3\) matrix and \(P\) is a \(3 \times 1\) matrix, so their product should be a \(4 \times 1\) matrix.

### Guided Practice

3. **BASKETBALL** Refer to the beginning of the lesson. Use matrix multiplication to determine in which season Lisa Leslie scored the most points. How many points did she score that season?

### Multiplicative Properties
Recall that the properties of real numbers also held true for matrix addition. However, some of these properties do not always hold true for matrix multiplication.
Example 4  Test of the Commutative Property

Find each product if $G = \begin{bmatrix} 1 & 3 & -5 \\ 4 & -2 & 0 \end{bmatrix}$ and $H = \begin{bmatrix} 2 & 3 \\ -2 & -8 & 1 \\ 7 \end{bmatrix}$.

a. $GH$

\[
GH = \begin{bmatrix} 1 & 3 \\ 4 & -2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ -2 & 8 \end{bmatrix} \\
= \begin{bmatrix} 2 - 6 - 5 & 3 - 24 - 35 \\ 8 + 4 + 0 & 12 + 16 + 0 \end{bmatrix} = \begin{bmatrix} -9 & -56 \\ 12 & 28 \end{bmatrix}
\]

b. $HG$

\[
HG = \begin{bmatrix} 2 & -3 \\ -2 & 8 \\ 1 & -7 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 & -5 \\ 4 & -2 & 0 \end{bmatrix} \\
= \begin{bmatrix} 2 + 12 & 6 - 6 - 10 + 0 \\ -2 - 32 & -6 + 16 + 10 + 0 \\ 1 + 28 & 3 - 14 - 5 + 0 \end{bmatrix} = \begin{bmatrix} 14 & 0 & -10 \\ -34 & 10 & 10 \\ 29 & -11 & -5 \end{bmatrix}
\]

Notice that $GH \neq HG$.

Example 4 demonstrates that the Commutative Property of Multiplication does not hold for matrix multiplication. The order in which you multiply matrices is very important.

Example 5  Test of the Distributive Property

Find each product if $J = \begin{bmatrix} 2 & 4 \\ -5 & -2 \end{bmatrix}$, $K = \begin{bmatrix} 3 & 2 \\ -1 & 3 \end{bmatrix}$, and $L = \begin{bmatrix} -4 & -1 \\ 3 & 0 \end{bmatrix}$.

a. $J(K + L)$

\[
J(K + L) = \begin{bmatrix} 2 & 4 \\ -5 & -2 \end{bmatrix} \cdot \left( \begin{bmatrix} 3 & 2 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} -4 & -1 \\ 3 & 0 \end{bmatrix} \right) \\
= \begin{bmatrix} 2 & 4 \\ -5 & -2 \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 \\ 2 & 3 \end{bmatrix} \\
= \begin{bmatrix} -2 + 8 & 2 + 12 \\ -5 - 4 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 14 \\ -5 & -6 \end{bmatrix}
\]

b. $JK + JL$

\[
JK + JL = \begin{bmatrix} 2 & 4 \\ -5 & -2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ -5 & -2 \end{bmatrix} \cdot \begin{bmatrix} -4 & -1 \\ 3 & 0 \end{bmatrix} \\
= \begin{bmatrix} 2(3) + 4(-1) & 2(2) + 4(3) \\ -5(3) + (-2)(-1) & -5(2) + (-2)(3) \end{bmatrix} + \begin{bmatrix} 2(-4) + 4(3) & 2(-1) + 4(0) \\ -5(-4) + (-2)(3) & -5(-1) + (-2)(0) \end{bmatrix} \\
= \begin{bmatrix} 2 & 16 \\ -13 & -16 \end{bmatrix} + \begin{bmatrix} 4 & -2 \\ 14 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 14 \\ 1 & 11 \end{bmatrix}
\]

Notice that $J(K + L) = JK + JL$.

Guided Practice 5. Use the matrices $R = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$, $S = \begin{bmatrix} 4 & 6 \\ -2 & 5 \end{bmatrix}$, and $T = \begin{bmatrix} -3 & 7 \\ -4 & 8 \end{bmatrix}$ to determine if $(S + T)R = SR + TR$. 
The previous example suggests that the Distributive Property is true for matrix multiplication. Some properties of matrix multiplication are shown below.

### Key Concept: Properties of Matrix Multiplication

For any matrices \( A, B, \) and \( C \) for which the matrix product is defined and any scalar \( k \), the following properties are true.

- **Associative Property of Matrix Multiplication** \((AB)C = A(BC)\)
- **Associative Property of Scalar Multiplication** \(k(AB) = (kA)B = A(kB)\)
- **Left Distributive Property** \(C(A + B) = CA + CB\)
- **Right Distributive Property** \((A + B)C = AC + BC\)

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### Check Your Understanding

#### Example 1
Determine whether each matrix product is defined. If so, state the dimensions of the product.

1. \( A_{2 \times 4} \cdot B_{4 \times 3} \)
2. \( C_{5 \times 4} \cdot D_{5 \times 4} \)
3. \( E_{8 \times 6} \cdot F_{6 \times 10} \)

#### Examples 2–3
Find each product, if possible.

4. \[
\begin{bmatrix}
2 & 1 \\
7 & -5
\end{bmatrix} \cdot 
\begin{bmatrix}
-6 & 3 \\
-2 & -4
\end{bmatrix}
\]
5. \[
\begin{bmatrix}
10 & -2 \\
-7 & 3
\end{bmatrix} \cdot 
\begin{bmatrix}
1 & 4 \\
5 & -2
\end{bmatrix}
\]
6. \[
\begin{bmatrix}
9 & -2 \\
-6 & 4
\end{bmatrix} \cdot 
\begin{bmatrix}
-2 & 4 \\
-6 & -7
\end{bmatrix}
\]
7. \[
\begin{bmatrix}
-9 \\
6
\end{bmatrix} \cdot 
\begin{bmatrix}
-1 & -10 \\
1 & 1
\end{bmatrix}
\]
8. \[
\begin{bmatrix}
-8 & 7 & 4 \\
-5 & -3 & 8
\end{bmatrix} \cdot 
\begin{bmatrix}
10 & 6 \\
8 & 4
\end{bmatrix}
\]
9. \[
\begin{bmatrix}
2 & 8 \\
3 & -1
\end{bmatrix} \cdot 
\begin{bmatrix}
6 \\
-7
\end{bmatrix}
\]
10. \[
\begin{bmatrix}
-4 & 3 & 2 \\
-1 & -5 & 4
\end{bmatrix} \cdot 
\begin{bmatrix}
2 & 1 & 6 \\
8 & 4 & -1 \\
5 & 3 & -2
\end{bmatrix}
\]
11. \[
\begin{bmatrix}
2 & 5 & 3 & -1 \\
-3 & 1 & 8 & -3
\end{bmatrix} \cdot 
\begin{bmatrix}
6 & -3 \\
-7 & 1 \\
2 & 0 \\
-1 & 0
\end{bmatrix}
\]

#### 12. FITNESS
The table shows the number of people registered for aerobics for the first quarter.

Quinn's Gym charges the following registration fees: class-by-class, $165; 11-class pass, $110; unlimited pass, $239.

- **a.** Write a matrix for the registration fees and a matrix for the number of students.
- **b.** Find the total amount of money the gym received from aerobics and step aerobic registrations.

#### Examples 4–5
Use \( X = \begin{bmatrix} -10 & -3 \\ 2 & -8 \end{bmatrix} \), \( Y = \begin{bmatrix} -5 & 6 \\ -1 & 9 \end{bmatrix} \), and \( Z = \begin{bmatrix} -5 & -1 \\ -8 & -4 \end{bmatrix} \) to determine whether the following equations are true for the given matrices.

13. \( XY = YX \)
14. \( X(YZ) = (XY)Z \)
**Examples 2–3** Find each product, if possible.

21. \( \begin{bmatrix} 1 & 6 \\ -3 & -7 \end{bmatrix} \cdot \begin{bmatrix} 4 & 4 \\ 9 & -3 \end{bmatrix} \) 

22. \( \begin{bmatrix} 6 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 2 & -7 \end{bmatrix} \)

23. \( \begin{bmatrix} -1 & 0 & 6 \\ -4 & -10 & 4 \end{bmatrix} \cdot \begin{bmatrix} 5 & -7 \\ 6 & 0 \end{bmatrix} \) 

24. \( \begin{bmatrix} -1 & 0 \\ 5 & 2 \end{bmatrix} \cdot \begin{bmatrix} 6 & -3 \\ 7 & -2 \end{bmatrix} \)

25. \( \begin{bmatrix} 2 & 9 & -3 \\ 4 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 4 & 2 \\ -6 & 7 \\ -2 & 1 \end{bmatrix} \) 

26. \( \begin{bmatrix} -6 & 4 & 2 \\ 2 & 8 & 7 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 2 \\ 4 \end{bmatrix} \)

27. \( \begin{bmatrix} 2 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix} \)

28. \( \begin{bmatrix} -4 \\ 8 \end{bmatrix} \cdot \begin{bmatrix} -3 & -1 \end{bmatrix} \)

**29. TRAVEL** The Wolf family owns three bed and breakfasts in a vacation spot. A room with a single bed is $220 a night, a room with two beds is $250 a night, and a suite is $360.

- **a.** Write a matrix for the number of each type of room at each bed and breakfast. Then write a room-cost matrix.

- **b.** Write a matrix for total daily income, assuming that all the rooms are rented.

- **c.** What is the total daily income from all three bed and breakfasts, assuming that all the rooms are rented?

**Available Rooms at a Wolf Bed and Breakfast**

<table>
<thead>
<tr>
<th>B &amp; B</th>
<th>Single</th>
<th>Double</th>
<th>Suite</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>

**Examples 4–5** Use \( P = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix} \), \( Q = \begin{bmatrix} 6 & 4 \\ -2 & -5 \end{bmatrix} \), \( R = \begin{bmatrix} 4 & 6 \\ -6 & 4 \end{bmatrix} \), and \( k = 2 \) to determine whether the following equations are true for the given matrices.

30. \( k(PQ) = P(kQ) \)

31. \( PQR = RQP \)

32. \( PR + QR = (P + Q)R \)

33. \( R(P + Q) = PR + QR \)

**34. FLOWERS** Student Council is selling flowers for Mother’s Day. They bought 200 roses, 150 daffodils, and 100 orchids for the purchase prices shown. They sold all of the flowers for the sales prices shown.

- **a.** Organize the data in two matrices, and use matrix multiplication to find the total amount that was spent on the flowers.

- **b.** Write two matrices, and use matrix multiplication to find the total amount the student council received for the flower sale.

- **c.** Use matrix operations to find how much money the student council made on their project.

<table>
<thead>
<tr>
<th>Flower</th>
<th>Purchase Price</th>
<th>Sales Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>rose</td>
<td>$1.67</td>
<td>$3.00</td>
</tr>
<tr>
<td>daffodil</td>
<td>$1.03</td>
<td>$2.25</td>
</tr>
<tr>
<td>orchid</td>
<td>$2.59</td>
<td>$4.50</td>
</tr>
</tbody>
</table>
35. AUTO SALES A car lot has four sales associates.
At the end of the year, each sales associate gets a bonus of $1000 for every new car they have sold and $500 for every used car they have sold.

a. Use a matrix to determine which sales associate earned the most money.
b. What is the total amount of money the car lot spent on bonuses for the sales associates this year?

44. CAMERAS Prices of digital cameras depend on features like optical zoom, digital zoom, and megapixels.

a. The 10-mp cameras are on sale for 20% off, and the other models are 10% off. Write a new matrix for these changes.
b. Write a new matrix allowing for a 6.25% sales tax on the discounted prices.
c. Describe what the differences in these two matrices represent.

45. BUSINESS The Kangy Studio has packages available for senior portraits.

a. Use matrices to determine the total cost of each package.
b. The studio offers an early bird discount of 15% off any package. Find the early bird price for each package.

H.O.T. Problems Use Higher-Order Thinking Skills

46. REASONING If the product matrix $AB$ has dimensions $5 \times 8$, and $A$ has dimensions $5 \times 6$, what are the dimensions of matrix $B$?

47. PROOF Show that each property of matrices is true for all $2 \times 2$ matrices.
   a. Scalar Distributive Property
   b. Matrix Distributive Property
   c. Associative Property of Multiplication
   d. Associative Property of Scalar Multiplication

48. OPEN ENDED Write two matrices $A$ and $B$ such that $AB = BA$.

49. CHALLENGE Find the missing values in $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 10 & 11 \\ 20 & 29 \end{bmatrix}$.

50. WRITING IN MATH Use the data on Lisa Leslie found at the beginning of the lesson to explain how matrices can be used in sports statistics. Describe a matrix that represents the total number of points she has scored during her career and an example of a sport in which different point values are used in scoring.
51. **GRIDDED RESPONSE** The average (arithmetic mean) of \( r, w, x, \) and \( y \) is 8, and the average of \( x \) and \( y \) is 4. What is the average of \( r \) and \( w \)?

52. Carla, Meiko, and Kayla went shopping to get ready for college. Their purchases and total amounts spent are shown in the table below.

<table>
<thead>
<tr>
<th>Person</th>
<th>Shirts</th>
<th>Pants</th>
<th>Shoes</th>
<th>Total Spent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Carla</td>
<td>3</td>
<td>4</td>
<td>2</td>
<td>$149.79</td>
</tr>
<tr>
<td>Meiko</td>
<td>5</td>
<td>3</td>
<td>3</td>
<td>$183.19</td>
</tr>
<tr>
<td>Kayla</td>
<td>6</td>
<td>5</td>
<td>1</td>
<td>$181.14</td>
</tr>
</tbody>
</table>

Assume that all of the shirts were the same price, all of the pants were the same price, and all of the shoes were the same price. What was the price of each item?

A shirt, $12.95; pants, $15.99; shoes, $23.49
B shirt, $15.99; pants, $12.95; shoes, $23.49
C shirt, $15.99; pants, $23.49; shoes, $12.95
D shirt, $23.49; pants, $15.99; shoes, $12.95

53. **GEOMETRY** Rectangle \( LMNQ \) has diagonals that intersect at point \( P \).

Which of the following represents point \( P \)?

- F (2, 2)
- H (0, 0)
- G (1, 1)
- J (−1, −1)

54. **SAT/ACT** What are the dimensions of the matrix that results from the multiplication shown?

\[
\begin{bmatrix}
a & b & c \\
d & e & f \\
g & h & i
\end{bmatrix}
\begin{bmatrix}
7 \\
4 \\
6
\end{bmatrix}
\]

A 1 × 4
B 3 × 3
C 3 × 1
D 4 × 1

55. \[
4 \begin{bmatrix}
8 & -1 \\
-3 & -4
\end{bmatrix}
- 5 \begin{bmatrix}
2 & -4 \\
6 & 3
\end{bmatrix}
\]

56. \[
5 \begin{bmatrix}
2 & -2 & -5 \\
-1 & 1 & 3
\end{bmatrix}
- 3 \begin{bmatrix}
-1 & -2 \\
6 & 4
\end{bmatrix}
\]

57. \[-4 \begin{bmatrix}
8 & 9 \\
-5 & 5
\end{bmatrix}
- 2 \begin{bmatrix}
-6 & -1 \\
6 & 3
\end{bmatrix}
\]

58. \[
\begin{bmatrix}
-2 & 1
\end{bmatrix}
\]

59. \[
\begin{bmatrix}
1 & 6 \\
-8 & -3
\end{bmatrix}
\]

60. \[
\begin{bmatrix}
9 & 1 \\
2 & 3 \\
5 & -3 \\
-9 & 0
\end{bmatrix}
\]

61. **MEDICINE** The graph shows how much Americans spent on doctors’ visits in some recent years and a prediction for 2014. (Lesson 2-5)

a. Find a regression equation for the data without the predicted value.

b. Use your equation to predict the expenditures for 2014.

c. Compare your prediction to the one given in the graph.

62. How many different ways can the letters of the word \( \text{MATHEMATICS} \) be arranged? (Lesson 0-5)

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- J (−1, −1)

54. **SAT/ACT** What are the dimensions of the matrix that results from the multiplication shown?

\[
\begin{bmatrix}
a & b & c \\
d & e & f \\
g & h & i
\end{bmatrix}
\begin{bmatrix}
7 \\
4 \\
6
\end{bmatrix}
\]

A 1 × 4
B 3 × 3
C 3 × 1
D 4 × 1

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a. Find a regression equation for the data without the predicted value.

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63. \( f(x) = |x - 4| + 3 \)

64. \( f(x) = 2|x + 3| - 5 \)

65. \( f(x) = (x + 2)^2 - 6 \)
State the dimensions of each matrix. (Lesson 4-1)

1. \[
\begin{bmatrix}
3 & 4 & 5 & 6 & 7
\end{bmatrix}
\]

2. \[
\begin{bmatrix}
10 & -6 & 18 & 0 \\
-7 & 5 & 2 & 4 \\
3 & 11 & 9 & 7
\end{bmatrix}
\]

Identify each element for the following matrices. (Lesson 4-1)

\[A = \begin{bmatrix}
4 & 3 \\
-5 & 1 \\
-3 & 7
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & -9 & 2 \\
0 & 10 & 4
\end{bmatrix}\]

3. \[a_{21}\]

4. \[b_{22}\]

5. **FUNDRAISER** The ninth and tenth grade classes sold different types of clothing to raise money. Two weeks of sales are shown in the table. (Lesson 4-2)

<table>
<thead>
<tr>
<th>Grade</th>
<th>Week</th>
<th>Type of Clothing</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>T-shirt</td>
</tr>
<tr>
<td>9</td>
<td>1</td>
<td>25</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>32</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>44</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>18</td>
</tr>
</tbody>
</table>

a. Write a matrix for each week’s sales.

b. Find the sum of the two weeks’ sales using matrix addition.

Perform the indicated operations. If the matrix does not exist, write impossible. (Lessons 4-2 and 4-3)

6. \[
\begin{bmatrix}
0 & 15 \\
-6 & -10
\end{bmatrix} - \begin{bmatrix}
8 & 0 \\
-3 & 5
\end{bmatrix}
\]

7. \[
-3 \begin{bmatrix}
3 & 5 & 12 \\
0 & -1 & 3 \\
9 & 6 & -5
\end{bmatrix}
\]

8. \[
\begin{bmatrix}
-1 \\
5 \\
-6
\end{bmatrix} + 4 \begin{bmatrix}
-3x \\
2 \\
x
\end{bmatrix} - 3 \begin{bmatrix}
x - 2 \\
3 \\
1
\end{bmatrix}
\]

9. **MULTIPLE CHOICE** Find \[
2 \begin{bmatrix}
3 & 5 \\
-6 & 0
\end{bmatrix} + 4 \begin{bmatrix}
9 & 2 \\
2 & 3
\end{bmatrix}.
\] (Lesson 4-2)

Find each product if possible. (Lesson 4-3)

10. \[
\begin{bmatrix}
-2 & 3 \\
1 & 6
\end{bmatrix} \cdot \begin{bmatrix}
3 & -1 & 4 \\
0 & 5 & -6
\end{bmatrix}
\]

11. \[
\begin{bmatrix}
-4 & 0 & -1 \\
0 & 1 & 8
\end{bmatrix} \cdot \begin{bmatrix}
-1 & 0 \\
0 & 4
\end{bmatrix}
\]

12. \[
\begin{bmatrix}
4 & -2 & -7 \\
6 & 3 & 5
\end{bmatrix} \cdot \begin{bmatrix}
-2 \\
5 \\
3
\end{bmatrix}
\]

13. **MULTIPLE CHOICE** If the product matrix \(XY\) has dimensions \(3 \times 2\) and \(X\) has dimensions \(3 \times 4\), what are the dimensions of matrix \(Y\)? (Lesson 4-3)

   - F \(2 \times 3\)
   - H \(3 \times 4\)
   - G \(3 \times 2\)
   - J \(4 \times 2\)

14. **SALES** Alex is in charge of stocking shirts for the concession stand at the high school football game. The number of shirts needed for a regular season game is listed in the matrix. Alex plans to double the number of shirts stocked for a playoff game. (Lesson 4-3)

<table>
<thead>
<tr>
<th>Size</th>
<th>small</th>
<th>medium</th>
<th>large</th>
</tr>
</thead>
<tbody>
<tr>
<td>Child</td>
<td>10</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>Adult</td>
<td>25</td>
<td>35</td>
<td>45</td>
</tr>
</tbody>
</table>

a. Write a matrix \(A\) to represent the regular season stock.

b. What scalar can be used to determine a matrix \(M\) to represent the new numbers? Find \(M\).

c. What is \(M - A\)? What does this represent in this situation?

15. **MULTIPLE CHOICE** What is the product of \[
\begin{bmatrix}
4 & 0 & -2 \\
2 & -1 & 3 \\
0 & 0 & 4
\end{bmatrix}
\]
and \[
\begin{bmatrix}
-2 & 0 \\
-3 & 0 \\
0 & 4
\end{bmatrix}?
\] (Lesson 4-3)

   - A \[
\begin{bmatrix}
8 & -12 \\
-12 & -12
\end{bmatrix}
\]
   - C \[
\begin{bmatrix}
8 & -4 \\
0 & 0 \\
0 & -8
\end{bmatrix}
\]
   - B \[
\begin{bmatrix}
8 \\
-12
\end{bmatrix}
\]
   - D Impossible

Determine whether each matrix product is defined. If so, state the dimensions of the product.

16. \(A_2 \times 3 \cdot B_3 \times 2\)
17. \(A_4 \times 1 \cdot B_2 \times 1\)
18. \(A_2 \times 5 \cdot B_5 \times 5\)
19. \(A_1 \times 5 \cdot B_5 \times 3\)
Transformations with Matrices

1 Translations and Dilations

Points on a coordinate plane can be represented by matrices. The ordered pair \((x, y)\) can be represented by the column matrix
\[
\begin{bmatrix}
  x \\
  y
\end{bmatrix}
\]
Likewise, polygons can be represented by placing all of the column matrices of the coordinates of the vertices into one matrix. This is called a vertex matrix or coordinate matrix.

Triangle \(ABC\) with vertices \(A(-4, -3)\), \(B(-2, 2)\), and \(C(3, -1)\) can be represented by the following vertex matrix.
\[
\Delta ABC = \begin{bmatrix}
  A & B & C \\
  -4 & -2 & 3 \\
  -3 & 2 & -1
\end{bmatrix}
\]

Matrices can be used to perform transformations. Transformations are functions that map points of a preimage onto its image.

Recall that one type of transformation is a translation. A translation occurs when a figure is moved from one location to another without changing its size, shape, or orientation. You can use matrix addition and a translation matrix to determine the coordinates of a translation image. The dimensions of a translation matrix should be the same as the dimensions of the vertex matrix.

- **Preimage**
- **Translation**
- **Image**

\[
\begin{bmatrix}
  -4 & -2 & 3 \\
  -3 & 2 & -1
\end{bmatrix}
+ \begin{bmatrix}
  2 & 2 & 2 \\
  -1 & -1 & -1
\end{bmatrix}
= \begin{bmatrix}
  -2 & 0 & 5 \\
  -4 & 1 & -2
\end{bmatrix}
\]
Example 1 Translation

Find the coordinates of the vertices of the image of quadrilateral $DEFG$ with $D(-4, -2)$, $E(0, -1)$, $F(2, -4)$, and $G(-2, -5)$ if it is translated 1 unit left and 4 units up. Then graph $DEFG$ and its image $D'E'F'G'$.

Write the vertex matrix for quadrilateral $DEFG$. \[
\begin{bmatrix}
-4 & 0 & 2 & -2 \\
-2 & -1 & -4 & -5
\end{bmatrix}
\]

To translate the quadrilateral 1 unit to the left, add -1 to each $x$-coordinate. To translate the figure 4 units up, add 4 to each $y$-coordinate. This can be done by adding the translation matrix \[
\begin{bmatrix}
-1 & -1 & -4 & -1 \\
4 & 4 & 4 & 4
\end{bmatrix}
\]
to the vertex matrix of $DEFG$.

\[
\begin{bmatrix}
-4 & 0 & 2 & -2 \\
-2 & -1 & -4 & -5
\end{bmatrix} + \begin{bmatrix}
-1 & -1 & -4 & -1 \\
4 & 4 & 4 & 4
\end{bmatrix} = \begin{bmatrix}
-5 & -1 & 1 & -3 \\
2 & 3 & 0 & -1
\end{bmatrix}
\]

The vertices of $D'E'F'G'$ are $D'(-5, 2)$, $E'(-1, 3)$, $F'(1, 0)$, and $G'(-3, -1)$.

$DEFG$ and $D'E'F'G'$ have the same size, shape, and orientation.

Guided Practice

1. Find the coordinates of the vertices of the image of triangle $RST$ with $R(-1, 5)$, $S(2, 1)$, and $T(-3, 2)$ if it is moved 3 units to the right and 4 units up. Then graph $RST$ and its image $R'S'T'$.

You can work backward to find a translation matrix.

Test Example 2

Rectangle $H'J'K'L'$ is an image of rectangle $HJKL$. A table of the vertices of each rectangle is shown. What are the coordinates of $K'$?

<table>
<thead>
<tr>
<th>HJKL</th>
<th>H'J'K'L'</th>
</tr>
</thead>
<tbody>
<tr>
<td>H(4, -5)</td>
<td>H'(-1, -8)</td>
</tr>
<tr>
<td>J(2, -8)</td>
<td>J'(-3, -11)</td>
</tr>
<tr>
<td>K(-4, -4)</td>
<td>K'(? , ?)</td>
</tr>
<tr>
<td>L(-2, -1)</td>
<td>L'(-7, -4)</td>
</tr>
</tbody>
</table>

A $(-5, -3)$ B $(-9, -7)$ C $(1, -1)$ D $(-7, -9)$

Read the Test Item

You are given the coordinates of the preimage and image of points $H$, $J$, and $L$. Use this information to find the translation matrix. Then you can use the translation matrix to find the coordinates of $K'$. 

Solve the Test Item

**Step 1** Write a matrix equation. Let \((a, b)\) represent the coordinates of \(K’\).

\[
\begin{bmatrix}
4 & -4 & -2 \\
-5 & -8 & -4
\end{bmatrix} + \begin{bmatrix}
x & x & x \\
y & y & y
\end{bmatrix} = \begin{bmatrix}
-1 & -3 & a & -7 \\
-8 & -11 & b & -4
\end{bmatrix}
\]

\[
\begin{bmatrix}
4 + x & 2 + x & -4 + x & -2 + x \\
-5 + y & -8 + y & -4 + y & -1 + y
\end{bmatrix} = \begin{bmatrix}
-1 & -3 & a & -7 \\
-8 & -11 & b & -4
\end{bmatrix}
\]

**Step 2** The matrices are equal, so corresponding elements are equal. Use the elements from the first row in each matrix.

\[
4 + x = -1 \quad \text{Solve for } x. \quad -5 + y = -8 \quad \text{Solve for } y.
\]

\[
x = -5 \quad y = -3
\]

**Step 3** Use the values for \(x\) and \(y\) to find the values for \(K'(a, b)\).

\[
-4 + (-5) = a \quad -4 + (-3) = b
\]

\[
-9 = a \quad -7 = b
\]

So, the coordinates of \(K’\) are \((-9, -7)\) and the answer is B. Check by solving different equations for \(x\) and \(y\).

**Guided Practice**

2. Triangle \(X’Y’Z’\) is the image of triangle \(XYZ\). Find the coordinates of \(Z’\) using the information shown in the table.

<table>
<thead>
<tr>
<th>Triangle XYZ</th>
<th>Triangle H'J'K'</th>
</tr>
</thead>
<tbody>
<tr>
<td>(X(3, -1))</td>
<td>(X'(1, 0))</td>
</tr>
<tr>
<td>(Y(-4, 2))</td>
<td>(Y'(-6, 3))</td>
</tr>
<tr>
<td>(Z(5, 1))</td>
<td>(Z'(?, ?))</td>
</tr>
</tbody>
</table>

G (7, 2)  J (3, 0)

A dilation is always performed relative to its center. Unless otherwise specified, the center is always the origin. You can use scalar multiplication to perform dilations.

**Example 3 Dilation**

Dilate \(\triangle MNP\) with \(M(-1, 1), N(2, 3),\) and \(P(1, -2)\) so that the perimeter of the image is twice that of the preimage. Find the coordinates of the vertices of \(\triangle M’N’P’\). Then graph \(\triangle MNP\) and \(\triangle M’N’P’\).

If the perimeter of the image is twice that of the preimage, then the lengths of the sides of the figure will be twice the measure of the original lengths. Multiply the vertex matrix by the scale factor 2.

\[
2 \begin{bmatrix}
-1 & 2 & 1 \\
1 & 3 & -2
\end{bmatrix} = \begin{bmatrix}
-2 & 4 & 2 \\
2 & 6 & -4
\end{bmatrix}
\]

The coordinates of the vertices of \(\triangle M’N’P’\) are \(M’(-2, 2), N’(4, 6),\) and \(P’(2, -4)\).

The preimage and image are similar. Both figures have the same shape.

**Guided Practice**

3. Dilate rectangle \(WXYZ\) with \(W(4, 4), X(4, 12), Y(8, 4),\) and \(Z(8, 12)\) so that the perimeter of the image is one fourth that of the preimage. Find the coordinates of the vertices of rectangle \(W’X’Y’Z’\).
Reflections and Rotations A reflection maps every point of a preimage to an image in a line of symmetry using a reflection matrix.

### KeyConcept Reflection Matrices

To reflect in the given line, multiply the vertex matrix by the given matrix.

<table>
<thead>
<tr>
<th>Line of Reflection</th>
<th>x-axis</th>
<th>y-axis</th>
<th>line $y = x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multiply on the left by:</td>
<td>$\begin{bmatrix} 1 &amp; 0 \ 0 &amp; -1 \end{bmatrix}$</td>
<td>$\begin{bmatrix} -1 &amp; 0 \ 0 &amp; 1 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 &amp; 1 \ 1 &amp; 0 \end{bmatrix}$</td>
</tr>
</tbody>
</table>

#### Example 4 Reflection

Find the coordinates of the vertices of the image of $QRST$ with $Q(3, 2), R(4, -3), S(-3, -4)$, and $T(-2, 1)$ after reflection in $y = x$. Graph the preimage and image.

Write the ordered pairs as a vertex matrix. Then multiply the vertex matrix by the reflection matrix for the line $y = x$.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 3 & 4 & -3 & -2 \\ 2 & -3 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -3 & -4 & 1 \\ 3 & 4 & -3 & -2 \end{bmatrix}$$

The coordinates of the vertices of quadrilateral $Q'R'S'T'$ are $Q'(2, 3), R'(-3, 4), S'(-4, -3)$, and $T'(1, -2)$.

#### Guided Practice

4. Find the coordinates of the vertices of the image of pentagon $ABCDE$ with $A(1, 3), B(3, 2), C(3, -1), D(1, -2)$, and $E(-1, 1)$ reflected across the $x$-axis.

A rotation maps every point of a preimage to an image rotated about a center point, usually the origin, using a rotation matrix.

### KeyConcept Rotation Matrices

To rotate counterclockwise about the origin, multiply the vertex matrix by the given matrix.

<table>
<thead>
<tr>
<th>Angle of Rotation</th>
<th>$90^\circ$</th>
<th>$180^\circ$</th>
<th>$270^\circ$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multiply on the left by:</td>
<td>$\begin{bmatrix} 0 &amp; -1 \ 1 &amp; 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} -1 &amp; 0 \ 0 &amp; -1 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 &amp; 1 \ -1 &amp; 0 \end{bmatrix}$</td>
</tr>
</tbody>
</table>

#### Study Tip

**Reflection** In a reflection, the preimage and image are congruent.

**Rotation** Unless otherwise indicated, all rotations in this text will be counterclockwise.
Rotation

In a rotation, like a reflection, the preimage and image are congruent.

Example 5  Rotation

Find the coordinates of the vertices of the image of \( \triangle VWX \) with \( V(3, 2), W(4, 1), \) and \( X(2, 1) \) after it is rotated 270° counterclockwise about the origin.

Write the ordered pairs in a vertex matrix. Then multiply the vertex matrix by the rotation matrix.

\[
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix} \cdot 
\begin{bmatrix}
3 & 4 & 2 \\
2 & 1 & 1
\end{bmatrix} = 
\begin{bmatrix}
2 & 1 & 1 \\
-3 & -4 & -2
\end{bmatrix}
\]

The coordinates of the vertices of \( \triangle V'W'X' \) are \( V'(2, -3), W'(1, -4), \) and \( X'(1, -2) \).

The image is congruent to the preimage. Both figures have the same size and shape.

Guided Practice

5. Find the coordinates of the vertices of the image of \( \triangle XYZ \) with \( X(-5, -6), \) \( Y(-1, -3), \) and \( Z(-2, -4) \) after it is rotated 180° counterclockwise about the origin.

Check Your Understanding

Example 1  Find the coordinates of the vertices of the image for each figure after the given translation. Then graph the preimage and image.

1. \( \triangle ABC \) with vertices \( A(-3, -3), B(-4, 2), \) and \( C(1, 0) \), translated 5 units right and down 3 units
2. quadrilateral \( WXYZ \) with vertices \( W(1, -3), X(0, 2), Y(1, 2), \) and \( Z(2, 1) \), translated 2 units left and up 4 units

Example 2  \( \text{MULTIPLE CHOICE} \) Rectangle \( RSTU \) with vertices \( R(-3, 2), S(1, 2), T(-3, -1), \) and \( U(1, -1) \) is translated so that \( T' \) is at \( (-4, 1) \). What are the coordinates of \( R' \) and \( U' \)?

A  \( R'(-4, 4), U'(0, 1) \)  C  \( R'(-2, 0), U'(0, -3) \)
B  \( R'(0, 1), U'(-4, 4) \)  D  \( R'(0, -2), U'(-3, 0) \)

Example 3  Find the coordinates of the vertices of the image after the given dilation. State the coordinates of the vertices of the image. Then graph the preimage and image.

4. \( \triangle DEF \) with vertices \( D(-2, -1), E(0, 3), \) and \( F(2, -1) \), dilated so that its perimeter is three times the original perimeter
5. square \( STUV \) with vertices \( S(1, 0), T(4, -3), U(1, -6), \) and \( V(-2, -3) \), dilated so that its perimeter is one half the original perimeter

Example 4  Find the coordinates of the vertices of the image of each figure after a reflection in the given axis of symmetry. Then graph the preimage and image.

6. rectangle \( GHJK \) with vertices \( G(1, 5), H(3, 4), J(0, -2), \) and \( K(-2, -1) \); \( x \)-axis
7. \( \triangle PQR \) with vertices \( P(-1, 2), Q(4, -4), \) and \( R(-1, -4) \); \( y = x \)
Example 5  Find the coordinates of the vertices of the image of each figure after the given rotation. Then graph the preimage and image.

8. \( \triangle LMN \) with vertices \( L(3, -5), M(1, 2) \), and \( N(3, 3) \); 180° about the origin

9. quadrilateral \( ABCD \) with \( A(-4, 1), B(-3, 4), C(0, 4) \), and \( D(1, 1) \); 90° about the origin

Practice and Problem Solving

Example 1  Find the coordinates of the vertices of the image of each figure after the given translation. Then graph the preimage and image.

10. \( \triangle MNO \) with vertices \( M(-7, 6), N(1, 7) \), and \( O(-3, 1) \), translated 2 units right and 6 units down

11. quadrilateral \( EFGH \) with vertices \( E(-4, 3), F(2, -2), G(-2, -4) \), and \( H(-3, -3) \), translated 4 units left and 1 unit up

12. rectangle \( PQRS \) with vertices \( P(-2, -3), Q(-2, 2), R(1, 2) \), and \( S(1, -3) \), translated 2 units left and 5 units down

13. \( \triangle JKL \) with vertices \( J(1, 4), K(2, 1) \), and \( L(-1, -2) \), translated 3 units right and 4 units down

14. \( \triangle ABC \) with vertices \( A(-1, 2), B(2, 4) \), and \( C(3, -2) \), translated 5 units left and 3 units down

15. square \( WXYZ \) with vertices \( W(3, 1), X(3, 5), Y(7, 5) \), and \( Z(7, 1) \), translated 2 units left and 4 units up

Example 2 16. Quadrilateral \( A'B'C'D' \) is the image after a translation of quadrilateral \( ABCD \). A table of the vertices of each quadrilateral is shown.

<table>
<thead>
<tr>
<th>Quadrilateral ( ABCD )</th>
<th>Quadrilateral ( A'B'C'D' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A(4, 8) )</td>
<td>( A(-1, -2) )</td>
</tr>
<tr>
<td>( B(11, -10) )</td>
<td>( B'(? , ?) )</td>
</tr>
<tr>
<td>( C(?, ?) )</td>
<td>( C'(-17, -14) )</td>
</tr>
<tr>
<td>( D(-4, 6) )</td>
<td>( D'(-9, -4) )</td>
</tr>
</tbody>
</table>

17. MAPS Camila looks at a map of a city she is visiting to find the way to a mall. Currently Camila is standing at an intersection with coordinates \((5.5, 7)\). She figures out that she must go 3 blocks east and 4 blocks north to get to the mall.

a. Write a translation matrix that Camila can use to find the coordinates of the mall.

b. Using the translation matrix, what are the coordinates of the mall?

Example 3  Find the coordinates of the vertices of the image after the given dilation. State the coordinates of the vertices of the image. Then graph the preimage and image.

18. \( \triangle TUV \) with vertices \( T(-3, -1), U(2, -1) \), and \( V(1, -5) \), dilated so that the perimeter of the image is twice that of the preimage

19. square \( DEFG \) with vertices \( D(0, 4), E(2, 2), F(0, 0) \), and \( G(-2, 2) \), dilated by a scale factor of 4

20. rectangle \( KLMN \) with vertices \( K(-3, 8), L(-3, 2), M(-6, 2) \), and \( N(-6, 8) \), dilated by a scale factor of \( \frac{1}{3} \)

21. \( \triangle QRS \) with vertices \( Q(6, 4), R(8, 0) \), and \( S(0, 1) \), dilated so that the perimeter of the image is one fourth that of the preimage
Example 4
Find the coordinates of the vertices of the image after a reflection in the given axis of symmetry. Then graph the preimage and image.
22. $\triangle ABC$ with vertices $A(9, -10), B(-5, -6),$ and $C(7, 7)$; y-axis
23. $\triangle DEF$ with vertices $D(7, 4), E(4, 0),$ and $F(-3, 2)$; x-axis
24. quadrilateral $GHJK$ with vertices $G(-5, 6), H(-2, 10), J(0, 8),$ and $K(-4, -4)$; $y = x$
25. square $LMNP$ with vertices $L(1, -2), M(-5, -1),$ $N(-4, 5),$ and $P(2, 4)$; y-axis
26. $\triangle QRS$ with vertices $Q(-2, 2), R(4, 2),$ and $S(2, -6)$; x-axis
27. $\triangle TUV$ with vertices $T(-4, 5), U(1, 3),$ and $V(-2, 0)$; $y = x$
28. ANIMATION Yori has plotted all of the locations for her cartoon on a coordinate plane. The vertices of the outline of a character’s house have coordinates $(–3, 5), (–3, 7), (5, 7),$ and $(5, 5).$ Yori decides that she wants to move all of the locations so that they are reflected across the y-axis. What will be the new coordinates of the house?

Example 5
Find the coordinates of the vertices of the image after the given rotation. Then graph the preimage and image.
29. $\triangle XYZ$ with vertices $X(1, 2), Y(1, 4),$ and $Z(5, 2)$; $180^\circ$ about the origin
30. quadrilateral $ABCD$ with vertices $A(-1, 1), B(-4, 1),$ $C(-6, 5),$ and $D(-3, 5)$; $270^\circ$ about the origin
31. rectangle $EFGH$ with vertices $E(3, 2), F(3, -4), G(2, -4),$ and $H(2, 2)$; $270^\circ$ about the origin
32. square $JKLM$ with vertices $J(-3, 3), K(1, 3),$ $L(1, -1),$ and $M(-3, -1)$; $90^\circ$ about the origin
33. quadrilateral $NPQR$ with vertices $N(-2, -1), P(-3, -5), Q(-6, -5),$ and $R(-6, -1)$; $90^\circ$ about the origin
34. $\triangle STU$ with vertices $S(-4, 2), T(1, 5),$ and $U(-2, -1)$; $180^\circ$ about the origin
35. RIDES The world’s tallest Ferris wheel, the Singapore Flyer, is 42 stories tall and 165 meters across. Jeremy and Nicole take a ride on it while on vacation. When they are at the top of the Ferris wheel, they have coordinates $(0, 165)$. Find their coordinates after the wheel has turned $90^\circ$ counterclockwise.

Find the coordinates of the vertices of the image after the given transformations.
36. reflected in $y = x$, then rotated $270^\circ$ about the origin
37. translated 4 units left and 2 units up, then reflected in the x-axis
38. rotated $90^\circ$ about the origin after being reflected in the $y$-axis
39. rotated $180^\circ$ about the origin
40. **ARCHITECTURE** On the blueprint for a new house, the coordinates of the corners of the garage are (1, 25), (1, 5), (31, 5), and (31, 25). If the scale of the blueprint is \(\frac{1}{48}\) that of the actual structure, find the coordinates of the garage when it is built.

41. **MULTIPLE REPRESENTATIONS** In this problem, you will investigate the results of multiple reflections.
   a. **Symbolic** Write the vertex matrices for \(\triangle ABC\) with vertices \(A(-2, 3), B(4, 0),\) and \(C(2, -5)\) and its image \(\triangle A'B'C'\) after reflection in the \(x\)-axis.
   b. **Graphical** Graph \(\triangle ABC\) and \(\triangle A'B'C'\).
   c. **Verbal** Make a conjecture about the reflection of \(\triangle A'B'C'\) in the \(x\)-axis. Perform the matrix multiplication to verify in conjecture.
   d. **Analytical** Find one matrix that could be used to reflect a triangle in the \(x\)-axis twice. How does the matrix relate to in observations?

### H.O.T. Problems Use Higher-Order Thinking Skills

42. **OPEN ENDED** A composite transformation is a transformation involving two or more transformations performed in sequence. Write a transformation matrix that could be used to perform a composite transformation involving dilation and rotation of a figure.

43. **REASONING** Determine whether the following statement is *sometimes*, *always*, or *never* true. Explain your reasoning.

   The image of a dilation is congruent to its preimage.

44. **CHALLENGE** Write a transformation matrix for each of the following.
   a. reflection in the line \(y = -x\)
   b. rotation 90° clockwise about the origin

45. **OPEN ENDED** Consider the point represented by the matrix \[
\begin{bmatrix}
3 \\
-2
\end{bmatrix}
\]. Give an example of a translation matrix and a reflection matrix that when applied separately to the point produce the same image point.

46. **WHICH ONE DOESN’T BELONG?** Determine which of the transformations is not the same as the others. Explain your reasoning.

   A
   
   B
   
   C
   
   D

47. **PROOF** Show that rotating \(\triangle ABC\) with vertices \(A(x_1, y_1), B(x_2, y_2),\) and \(C(x_3, y_3)\) 180° counterclockwise about the origin is the same as reflecting the figure in the \(x\)-axis, then in the \(y\)-axis.

48. **WRITING IN MATH** Use the information in this lesson to describe how matrices can be used to transform figures in two-dimensional space. Describe the physical characteristics of a figure after each transformation.
49. Tonya wanted to find 5 consecutive whole numbers that add up to 95. She wrote the equation \((n - 2) + (n - 1) + n + (n + 1) + (n + 2) = 95\). What does the variable \(n\) represent in the equation?
   - A the least of the 5 whole numbers
   - B the middle of the 5 whole numbers
   - C the greatest of the 5 whole numbers
   - D the difference between the least and the greatest of the 5 whole numbers

50. PROBABILITY There are 12 songs on your MP3 player. The player is set to shuffle, but not repeat—that is, to play the songs in random order without repeating any. What is the probability that any two of the four country songs will be the first two played?
   - F \(P \approx 20\%\)
   - G \(P \approx 11.1\%\)
   - H \(P \approx 9.09\%\)
   - J \(P \approx 0.83\%

51. EXTENDED RESPONSE The following types of vehicles are rented at The Auto Store. Organize the data into a \(4 \times 2\) matrix. Then write a new matrix providing the prices after a 15% markup.

<table>
<thead>
<tr>
<th></th>
<th>Daily</th>
<th>Weekly</th>
</tr>
</thead>
<tbody>
<tr>
<td>Economy</td>
<td>$29.99</td>
<td>$149.99</td>
</tr>
<tr>
<td>Mid-Size</td>
<td>$39.99</td>
<td>$179.99</td>
</tr>
<tr>
<td>Full-Size</td>
<td>$49.99</td>
<td>$209.99</td>
</tr>
<tr>
<td>SUV</td>
<td>$69.99</td>
<td>$349.99</td>
</tr>
</tbody>
</table>

52. SAT/ACT Triangle \(ABC\) has vertices \(A(-4, 2), B(-4, -3),\) and \(C(3, -2)\). After a dilation, triangle \(A'B'C'\) has vertices \(A'(-12, 6), B'(-12, -9),\) and \(C'(9, -6)\). How many times as great is the perimeter of \(\triangle A'B'C'\) as that of \(\triangle ABC\)?
   - A \(\frac{1}{3}\)
   - B \(3\)
   - C \(6\)
   - D \(9\)
   - E \(12\)

53. \[
\begin{bmatrix}
4 & 2 \\
-1 & -3
\end{bmatrix}
\begin{bmatrix}
6 & 2 \\
5 & 1
\end{bmatrix}
\]
54. \[
\begin{bmatrix}
8 & -2 \\
-4 & -5
\end{bmatrix}
\begin{bmatrix}
-2 \\
3
\end{bmatrix}
\]
55. \[
\begin{bmatrix}
-3 \\
4
\end{bmatrix}
\begin{bmatrix}
-6 & -8 \\
-4 & 5
\end{bmatrix}
\]

56. BUSINESS The table lists the prices at the Sandwich Shoppe. (Lesson 4-2)
   a. List the prices in a \(4 \times 3\) matrix.
   b. The manager decides to cut the prices of every item by 20%. List this new set of data in a \(4 \times 3\) matrix.
   c. Subtract the second matrix from the first and determine the savings to the customer for each sandwich.

<table>
<thead>
<tr>
<th></th>
<th>Small</th>
<th>Medium</th>
<th>Large</th>
</tr>
</thead>
<tbody>
<tr>
<td>ham</td>
<td>$4.50</td>
<td>$6.75</td>
<td>$9.50</td>
</tr>
<tr>
<td>salami</td>
<td>$4.50</td>
<td>$6.75</td>
<td>$9.50</td>
</tr>
<tr>
<td>veggie</td>
<td>$4.00</td>
<td>$6.25</td>
<td>$8.75</td>
</tr>
<tr>
<td>meatball</td>
<td>$4.75</td>
<td>$7.50</td>
<td>$10.25</td>
</tr>
</tbody>
</table>

57. \(|2x + 5| + 3 \geq y\)
58. \(y \leq 2|x - 4|\)
59. \(y \geq -2|\ x - 3 | + 1\)

State whether the events are independent or dependent. (Lesson 0-4)

60. choosing a president, vice president, secretary, and treasurer for Key Club, assuming that a person can hold only one office
61. selecting a horror movie and a comedy at the video store

Skills Review

Solve each system of equations. (Lesson 3-2)
62. \(y = 3x - 10\)
   \(4x - 3y = 20\)
63. \(4y + 5x = 21\)
   \(2x + 7y = 3\)
64. \(-5x - 2y = 27\)
   \(8x + 5y = -54\)
A $1 \times 2$ or $2 \times 1$ matrix can be represented by a vector. A **vector** is a quantity that has both **magnitude**, or length, and **direction**. They are represented as directed segments such as $\overrightarrow{AB}$, read vector $AB$.

A vector on a coordinate grid can be described in **component form**, or in terms of its horizontal and vertical components. The notation for a vector in component form is $(x \text{ component}, y \text{ component})$. The vector $(2, 3)$ has initial point $(0, 0)$ and terminal point $(2, 3)$.

Matrix operations can be used to determine the component form of a vector.

**Example 1 Determine Component Form**

If $A = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$, express $\overrightarrow{AB}$ in component form. Then graph $\overrightarrow{AB}$.

**Step 1** Determine the $x$- and $y$-components by finding $B - A$.

$$B - A = \begin{bmatrix} 4 \\ 3 \end{bmatrix} - \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 - (-2) \\ 3 - 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

**Step 2** Express $\overrightarrow{AB}$ in component form.

$$\begin{bmatrix} 6 \\ 2 \end{bmatrix} \leftarrow x \text{ component} \quad \begin{bmatrix} \end{bmatrix} \leftarrow y \text{ component} \quad (6, 2)$$

**Step 3** Graph $\overrightarrow{AB}$.

Because $\overrightarrow{AB} = (6, 2)$, begin the vector at $(0, 0)$ and extend it to $(6, 2)$. 
Example 2  Add Vectors

Find \( \overrightarrow{CD} + \overrightarrow{GH} \) for \((-2, -2), D(1, 3), G(2, 5), \) and \(H(4, -1)\).

**Step 1**  Express the ordered pairs as matrices.
\[
C = \begin{bmatrix} -2 \\ -2 \end{bmatrix}, \quad D = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad G = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \quad H = \begin{bmatrix} 4 \\ -1 \end{bmatrix}
\]

**Step 2**  Use matrix operations to represent each vector.
\[
\overrightarrow{CD}: D - C = \begin{bmatrix} 1 - (-2) \\ 3 - (-2) \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \quad \overrightarrow{GH}: H - G = \begin{bmatrix} 4 - 2 \\ -1 - 5 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}
\]

**Step 3**  Add the matrices to represent \( \overrightarrow{CD} + \overrightarrow{GH} \).
\[
\begin{bmatrix} 3 \\ 5 \end{bmatrix} + \begin{bmatrix} 2 \\ -6 \end{bmatrix} = \begin{bmatrix} 3 + 2 \\ 5 + (-6) \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}
\]

**Step 4**  Write the matrix as a component vector.
\[
\begin{bmatrix} 5 \\ -1 \end{bmatrix} \rightarrow (5, -1)
\]
Thus, \( \overrightarrow{CD} + \overrightarrow{GH} = (5, -1) \).

Exercises

Express \( \overrightarrow{XY} \) in component form. Then graph \( \overrightarrow{XY} \).

1. \( X = \begin{bmatrix} 5 \\ 9 \end{bmatrix}, \quad Y = \begin{bmatrix} -10 \\ -2 \end{bmatrix} \)
2. \( X = \begin{bmatrix} 8 \\ 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 3 \\ -5 \end{bmatrix} \)
3. \( X = \begin{bmatrix} -8 \\ -9 \end{bmatrix}, \quad Y = \begin{bmatrix} 8 \\ 9 \end{bmatrix} \)
4. \( X = \begin{bmatrix} -4 \\ -7 \end{bmatrix}, \quad Y = \begin{bmatrix} -4 \\ 6 \end{bmatrix} \)

Find \( \overrightarrow{PQ} + \overrightarrow{RS} \).

5. \( P(-1, -3), \quad Q(8, -6), \quad R(-5, -7), \quad S(4, 2) \)
6. \( P(0, 10), \quad Q(-5, 2), \quad R(-5, 0), \quad S(3, 4) \)
7. \( P(1, -6), \quad Q(-9, 0), \quad R(6, -6), \quad S(1, 9) \)
8. \( P(5, -8), \quad Q(2, 5), \quad R(-6, 2), \quad S(-7, 3) \)

Vectors can be subtracted by using the same method as addition. Find \( \overrightarrow{JK} - \overrightarrow{LM} \).

9. \( J(7, -1), \quad K(-3, 9), \quad L(-9, 4), \quad M(6, 0) \)
10. \( J(6, 7), \quad K(10, -4), \quad L(-2, 8), \quad M(3, 1) \)


12. **WRITING IN MATH** Explain how you could apply scalar multiplication of matrices to vectors. Describe how a vector is affected by scalar multiplication.
Determinants and Cramer’s Rule

1 **Determinants** Every square matrix has a determinant. The determinant of a $2 \times 2$ matrix is called a second-order determinant.

### Key Concept Second-Order Determinant

- **Words** The value of a second-order determinant is the difference of the products of the two diagonals.
- **Symbols**
  \[
  \text{det} \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a \\ c \end{vmatrix} \begin{vmatrix} d \end{vmatrix} = ad - bc
  \]
- **Example**
  \[
  \begin{vmatrix} 4 & 5 \\ -3 & 6 \end{vmatrix} = 4(6) - (-3)(5) = 39
  \]

### Example 1 Second-Order Determinant

Evaluate each determinant.

a. \[
\begin{vmatrix} 5 & -4 \\ 8 & 9 \end{vmatrix}
\]
\[
\begin{vmatrix} 5 & -4 \\ 8 & 9 \end{vmatrix} = 5(9) - 8(-4) \quad \text{Definition of determinant}
\]
\[
= 45 + 32 \quad \text{Simplify.}
\]
\[
= 77
\]

b. \[
\begin{vmatrix} 0 & 6 \\ 4 & -11 \end{vmatrix}
\]
\[
\begin{vmatrix} 0 & 6 \\ 4 & -11 \end{vmatrix} = 0(-11) - 4(6) \quad \text{Definition of determinant}
\]
\[
= 0 - 24 \quad \text{Simplify.}
\]
\[
= -24
\]

### Guided Practice

1A. \[
\begin{vmatrix} -6 & -7 \\ 10 & 8 \end{vmatrix}
\]

1B. \[
\begin{vmatrix} 7 & 5 \\ 9 & -4 \end{vmatrix}
\]

---

**Tennessee Curriculum Standards**

✓ 3103.3.8 Solve a three by three system of linear equations algebraically and by using inverse matrices and determinants with and without technology.

**Then**
- You solved systems of equations algebraically. (Lesson 3-2)

**Now**
- Evaluate determinants.
- Solve systems of linear equations by using Cramer’s Rule.

**Why?**
- A zoologist tagged a tiger with a GPS tracker so that she could determine the tiger’s territory. After several days, the zoologist determined that the tiger’s territory was a triangular region. By using the coordinates of the vertices of this triangle, she could use matrices and determinants to determine the size of the tiger’s territory.

---

**New Vocabulary**
- determinant
- second-order determinant
- third-order determinant
- diagonal rule
- Cramer’s Rule
- coefficient matrix

---

220 | Lesson 4-5
Study Tip

Diagonal Rule: The diagonal rule can only be used for $3 \times 3$ matrices.

Determinants of $3 \times 3$ matrices are called third-order determinants. They can be evaluated by using the diagonal rule.

Key Concept: Diagonal Rule

Step 1: Rewrite the first two columns to the right of the determinant.

Step 2: Draw diagonals, beginning with the upper left-hand element. Multiply the elements in each diagonal. Repeat the process, beginning with the upper right-hand element.

Step 3: Find the sum of the products of the elements in each set of diagonals.

Step 4: Subtract the second sum from the first sum.

Example 2: Use Diagonals

Evaluate

\[
\begin{vmatrix}
4 & -8 & 3 \\
-3 & 2 & 6 \\
-4 & 5 & 9
\end{vmatrix}
\]

using diagonals.

Step 1: Rewrite the first two columns to the right of the determinant.

\[
\begin{array}{ccc|ccc}
4 & -8 & 3 & 4 & -8 & 3 \\
-3 & 2 & 6 & -3 & 2 & 6 \\
-4 & 5 & 9 & -4 & 5 & 9 \\
\end{array}
\]

Step 2: Find the products of the elements of the diagonals.

\[
\begin{array}{ccc|ccc}
4 & -8 & 3 & 4 & -8 & 3 \\
-3 & 2 & 6 & -3 & 2 & 6 \\
-4 & 5 & 9 & -4 & 5 & 9 \\
\end{array}
\]

\[
4(2)(9) = 72 \\
-8(6)(-4) = 192 \\
3(-3)(5) = -45
\]

\[
\begin{array}{ccc|ccc}
4 & -8 & 3 & 4 & -8 & 3 \\
-3 & 2 & 6 & -3 & 2 & 6 \\
-4 & 5 & 9 & -4 & 5 & 9 \\
\end{array}
\]

\[
-4(2)(3) = -24 \\
5(6)(4) = 120 \\
9(-3)(-8) = 216
\]

Step 3: Find the sum of each group.

\[
72 + 192 + (-45) = 219 \\
-24 + 120 + 216 = 312
\]

Step 4: Subtract the sum of the second group from the sum of the first group.

\[
219 - 312 = -93
\]

The value of the determinant is $-93$.

Guided Practice

Evaluate each determinant.

2A.

\[
\begin{vmatrix}
-5 & 9 & 4 \\
-2 & -1 & 5 \\
-4 & 6 & 2
\end{vmatrix}
\]

2B.

\[
\begin{vmatrix}
-8 & -4 & 4 \\
0 & -5 & -8 \\
3 & 4 & 1
\end{vmatrix}
\]
Determinants can also be used to find the areas of triangles. If the coordinates of the vertices of the triangle are known, the formula below can be used to calculate the area of the triangle.

**Key Concept Area of a Triangle**

Words: The area of a triangle with vertices \((a, b), (c, d),\) and \((e, f)\) is \(A = \frac{1}{2} \left| \begin{array}{ccc} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{array} \right|,\)

Example: \(A = \frac{1}{2} \left| \begin{array}{ccc} -4 & 3 & 1 \\ 3 & 1 & 1 \\ -2 & -2 & 1 \end{array} \right|.\)

**Real-World Example 3 Use Determinants**

**ZOOLOGY** Refer to the application at the beginning of the lesson. The coordinates of the vertices of the tiger's territory are shown to the right. Use determinants to find the area of the tiger's territory.

\[ A = \frac{1}{2} \left| \begin{array}{ccc} 0 & 4 & -2 \\ 4 & 12 & 8 \\ -2 & 8 & 1 \end{array} \right| \]

Diagonal Rule

\[ \sum \text{products of diagonals} \]

\[ 0 + 0 + 32 = 32 \quad -24 + 0 + 0 = -24 \]

Area of a Triangle

\[ A = \frac{1}{2} \left| \begin{array}{ccc} 0 & 0 & 1 \\ 4 & 12 & 1 \\ -2 & 8 & 1 \end{array} \right| \]

\[ = \left( \frac{1}{2} \right) [32 - (-24)] = 28 \]

The area of the tiger’s territory is 28 square kilometers.

**Guided Practice**

3. **CAR WASH** To raise money for their rowing club, Hannah, Christina, and Dario are advertising a car wash at three different street corners in a neighborhood. On a map, the coordinates for the corners are \((3, 15), (6, 4),\) and \((11, 9).\) Each unit represents 0.5 kilometer. What is the area of the neighborhood in which they are advertising?
2 **Cramer's Rule**  You can use determinants to solve systems of equations. If a determinant is nonzero, then the system has a unique solution. If a determinant is 0, then the system either has no solution or infinite solutions. A method called **Cramer's Rule** uses the coefficient matrix. The **coefficient matrix** is a matrix that contains only the coefficients of the system.

### Key Concept: Cramer's Rule

Let $C$ be the coefficient matrix of the system $ax + by = m$

$\begin{array}{lcl}
fx + gy &=& n \\
\end{array}$

\[
C = \begin{bmatrix}
a & b \\
f & g \\
\end{bmatrix}
\]

The solution of this system is $x = \frac{\begin{vmatrix}
m & b \\
\end{vmatrix}}{|C|}$ and $y = \frac{\begin{vmatrix}
a & m \\
\end{vmatrix}}{|C|}$, if $C \neq 0$.

### Example 4: Solve a System of Two Equations

**Solve the system by using Cramer's Rule.**

$5x - 6y = 15$

$3x + 4y = -29$

$C = \begin{bmatrix}
15 & -6 \\
-29 & 4 \\
5 & -6 \\
3 & 4 \\
\end{bmatrix}$

$x = \frac{\begin{vmatrix}
m & b \\
\end{vmatrix}}{|C|}$

Cramer's Rule

$y = \frac{\begin{vmatrix}
a & m \\
\end{vmatrix}}{|C|}$

### Example 4: Solve a System of Two Equations

$C = \begin{bmatrix}
15 & -6 \\
-29 & 4 \\
5 & -6 \\
3 & 4 \\
\end{bmatrix}$

Evaluate.

$\frac{15(4) - (-29)(-6)}{5(4) - (3)(-6)}$  

$\frac{60 - 174}{20 + 18}$

$\frac{-114}{38}$

Simplify.

$\frac{-190}{38}$

$x = \frac{\begin{vmatrix}
m & b \\
\end{vmatrix}}{|C|}$

Substitute values.

$y = \frac{\begin{vmatrix}
a & m \\
\end{vmatrix}}{|C|}$

$\begin{bmatrix}
5 & 15 \\
3 & -29 \\
5 & -6 \\
3 & 4 \\
\end{bmatrix}$

Multiply.

$\frac{5(-29) - 3(15)}{5(4) - (3)(-6)}$

$\frac{-145 - 45}{20 + 18}$

$\frac{-190}{38}$

$\frac{-38}{38}$

$\frac{-3}{1}$

$\frac{5(-3) - 6(-5)}{15}$

$\frac{-15 + 30}{15}$

$\frac{15}{15}$

$\frac{3(-3) + 4(-5)}{-29}$

$\frac{-9 - 20}{-29}$

$\frac{-29}{-29}$

The solution of the system is $(-3, -5)$.

**CHECK**

$5(-3) - 6(-5) \neq 15$  

$x = -3, y = -5$

Simplify.

$-15 + 30 \neq 15$

$15 = 15$

$3(-3) + 4(-5) \neq -29$

$x = -3, y = -5$

Simplify.

$-9 - 20 \neq -29$

$-29 = -29$

**Guided Practice**

### Guided Practice

**Solve each system using Cramer’s Rule.**

4A. $7x + 3y = 37$

$-5x - 7y = -41$

4B. $8x - 5y = 70$

$9x + 7y = 3$
Cramer’s Rule can also be used for systems of three equations.

**Key Concept**  Cramer’s Rule for a System of Three Equations

Let \( C \) be the coefficient matrix of the system

\[
\begin{align*}
ax + by + cz &= m \\
f(x + gy + hz) &= n \\
jx + ky + \ell z &= p
\end{align*}
\]

The solution of this system is

\[
\begin{align*}
x &= \frac{\begin{vmatrix} m & b & c \\ n & g & h \\ p & k & \ell \end{vmatrix}}{|C|}, \\
y &= \frac{\begin{vmatrix} a & m & c \\ f & n & h \\ j & p & \ell \end{vmatrix}}{|C|}, \\
z &= \frac{\begin{vmatrix} a & b & m \\ f & g & n \\ j & k & \ell \end{vmatrix}}{|C|},
\end{align*}
\]

if \( C \neq 0 \).

**Example 5**  Solve a System of Three Equations

Solve the system by using Cramer’s Rule.

\[
\begin{align*}
4x + 5y - 6z &= -14 \\
3x - 2y + 7z &= 47 \\
7x - 6y - 8z &= 15
\end{align*}
\]

\[
\begin{align*}
x &= \frac{\begin{vmatrix} -14 & 5 & -6 \\ 47 & -2 & 7 \\ 15 & -6 & -8 \end{vmatrix}}{621} = \frac{3105}{621} \\
y &= \frac{\begin{vmatrix} 4 & -14 & -6 \\ 3 & 47 & 7 \\ 7 & 15 & -8 \end{vmatrix}}{621} = \frac{-1242}{621} \\
z &= \frac{\begin{vmatrix} 4 & 5 & -14 \\ 3 & -2 & 47 \\ 7 & -6 & 15 \end{vmatrix}}{621} = \frac{2484}{621}
\end{align*}
\]

The solution of the system is \((5, -2, 4)\).

**CHECK**

\[
\begin{align*}
&4(5) + 5(-2) - 6(4) = -14 && 3(5) - 2(-2) + 7(4) = 47 \\
&20 - 10 - 24 = -14 && 15 + 4 + 28 = 47 \\
&-14 = -14 \checkmark && 47 = 47 \checkmark
\end{align*}
\]

\[
\begin{align*}
&7(5) - 6(-2) - 8(4) = 15 \\
&35 + 12 - 32 = 15 \\
&15 = 15 \checkmark
\end{align*}
\]

**Guided Practice**

Solve each system using Cramer’s Rule.

5A. \(3x + 5y + 2z = -7\) \hspace{1cm} 5B. \(6x + 5y + 2z = -1\)

\[
\begin{align*}
&-4x + 3y - 5z = -19 \\
&5x + 4y - 7z = -15
\end{align*}
\]

\[
\begin{align*}
&-x + 3y + 7z = 12 \\
&5x - 7y - 3z = -52
\end{align*}
\]
**Example 1**

Evaluate each determinant.

1. \[
\begin{vmatrix}
8 & 6 \\
5 & 7
\end{vmatrix}
\]

2. \[
\begin{vmatrix}
-6 & -6 \\
8 & 10
\end{vmatrix}
\]

3. \[
\begin{vmatrix}
-4 & 12 \\
9 & 5
\end{vmatrix}
\]

4. \[
\begin{vmatrix}
16 & -10 \\
-8 & 5
\end{vmatrix}
\]

**Example 2**

Evaluate each determinant using diagonals.

5. \[
\begin{vmatrix}
3 & -2 & 2 \\
-4 & 2 & -5 \\
-3 & 1 & 4
\end{vmatrix}
\]

6. \[
\begin{vmatrix}
2 & -3 & 5 \\
-4 & 6 & -2 \\
4 & -1 & -6
\end{vmatrix}
\]

7. \[
\begin{vmatrix}
8 & 4 & 0 \\
-2 & -6 & -1 \\
5 & -3 & 6
\end{vmatrix}
\]

8. \[
\begin{vmatrix}
-5 & -3 & 4 \\
-2 & -4 & -3 \\
8 & 2 & 4
\end{vmatrix}
\]

9. \[
\begin{vmatrix}
2 & 4 & 2 \\
1 & 6 & 5
\end{vmatrix}
\]

10. \[
\begin{vmatrix}
1 & 5 & -2 \\
-1 & -8 & -3
\end{vmatrix}
\]

11. \[
\begin{vmatrix}
2 & -6 & -3 \\
7 & 9 & -4 \\
-6 & 4 & 9
\end{vmatrix}
\]

12. \[
\begin{vmatrix}
-5 & -6 & 7 \\
4 & 0 & 5 \\
-3 & 8 & 2
\end{vmatrix}
\]

**Example 4**

Use Cramer’s Rule to solve each system of equations.

13. \[4x - 5y = 39 \\
3x + 8y = -6 \]

14. \[5x + 6y = 20 \\
-3x - 7y = -29 \]

15. \[-8a - 5b = -27 \\
7a + 6b = 22 \]

16. \[10c - 7d = -59 \\
6c + 5d = -63 \]

**Examples 3–5**

17. **GEOGRAPHY**

The “Bermuda Triangle” is an area located off the southeastern Atlantic coast of the United States, and noted for reports of unexplained losses of ships, small boats, and aircraft.

a. Find the area of the triangle on the map.

b. Suppose each grid represents 175 miles. What is the area of the Bermuda Triangle?

**Use Cramer’s Rule to solve each system of equations.**

18. \[4x - 2y + 7z = 26 \\
5x + 3y - 5z = -50 \\
-7x - 8y - 3z = 49 \]

19. \[-3x - 5y + 10z = -4 \\
-8x + 2y - 3z = -91 \\
6x + 8y - 7z = -35 \]

20. \[6x - 5y + 2z = -49 \\
-5x - 3y - 8z = -22 \\
-3x + 8y - 5z = 55 \]

21. \[-9x + 5y + 3z = 50 \\
7x + 8y - 2z = -60 \\
-5x + 7y + 5z = 46 \]

22. \[x + 2y = 12 \\
3y - 4z = 25 \\
x + 6y + z = 20 \]

23. \[9a + 7b = -30 \\
8b + 5c = 11 \\
-3a + 10c = 73 \]

24. \[2n + 3p - 4w = 20 \\
4n - p + 5w = 13 \\
3n + 2p + 4w = 15 \]

25. \[x + y + z = 12 \\
6x - 2y - z = 16 \\
3x + 4y + 2z = 28 \]
Examples 1–2 Evaluate each determinant.

26. \[
\begin{vmatrix}
-7 & 12 \\
5 & 6 
\end{vmatrix}
\]

27. \[
\begin{vmatrix}
-8 & -9 \\
11 & 12 
\end{vmatrix}
\]

28. \[
\begin{vmatrix}
-5 & 8 \\
-6 & -7 
\end{vmatrix}
\]

29. \[
\begin{vmatrix}
3 & 5 & -2 \\
-1 & -4 & 6 \\
-6 & -2 & 5 
\end{vmatrix}
\]

30. \[
\begin{vmatrix}
2 & 0 & -6 \\
-3 & -4 & -5 \\
-2 & 5 & 8 
\end{vmatrix}
\]

31. \[
\begin{vmatrix}
-5 & -1 & -2 \\
1 & 8 & 4 \\
0 & -6 & 9 
\end{vmatrix}
\]

32. \[
\begin{vmatrix}
6 & -3 & -5 \\
0 & -7 & 0 \\
3 & -6 & -4 
\end{vmatrix}
\]

33. \[
\begin{vmatrix}
-8 & -3 & -9 \\
0 & 0 & 0 \\
8 & -2 & -4 
\end{vmatrix}
\]

34. \[
\begin{vmatrix}
1 & 6 & 7 \\
-2 & -5 & -8 \\
4 & 4 & 9 
\end{vmatrix}
\]

35. \[
\begin{vmatrix}
1 & -8 & -9 \\
6 & 5 & -6 \\
-2 & -8 & 10 
\end{vmatrix}
\]

36. \[
\begin{vmatrix}
5 & -5 & -5 \\
-8 & -3 & -2 \\
-2 & 4 & 6 
\end{vmatrix}
\]

37. \[
\begin{vmatrix}
-4 & 1 & -2 \\
10 & 12 & 9 \\
-6 & 0 & 13 
\end{vmatrix}
\]

38. TRAVEL Mr. Smith’s art class took a bus trip to an art museum. The bus averaged 65 miles per hour on the highway and 25 miles per hour in the city. The art museum is 375 miles away from the school, and it took the class 7 hours to get there. Use Cramer’s Rule to find how many hours the bus was on the highway and how many hours it was driving in the city.

Examples 4–5 Use Cramer’s Rule to solve each system of equations.

39. \[
\begin{align*}
6x - 5y &= 73 \\
-7x + 3y &= -71 \\
-4c - 5d &= -39 \\
5c + 8d &= 54 \\
9r + 4s &= -55 \\
-5r - 3s &= 36 \\
5x - 4y + 6z &= 58 \\
-4x + 6y + 3z &= -13 \\
6x + 3y + 7z &= 53 \\
\end{align*}
\]

40. \[
\begin{align*}
10a - 3b &= -34 \\
3a + 8b &= -28 \\
-6f - 8g &= -22 \\
-11f + 5g &= -60 \\
-11u - 7v &= 4 \\
9u + 4v &= -24 \\
8x - 4y + 7z &= 34 \\
5x + 6y + 3z &= -21 \\
3x + 7y - 8z &= -85 \\
\end{align*}
\]

47. DOUGHNUTS Mi-Ling is ordering doughnuts for a class party. The box contains 2 dozen doughnuts, some of which are plain and some of which are jelly-filled. The plain doughnuts each cost $0.50, and the jelly-filled doughnuts each cost $0.60. If the total cost is $12.60, use Cramer’s Rule to find how many jelly-filled doughnuts Mi-Ling ordered.

48. MOVIES The salary for each of the stars of a new movie is $5 million, and the supporting actors each receive $1 million. The total amount spent for the salaries of the actors and actresses is $19 million. If the cast has 7 members, use Cramer’s Rule to find the number of stars in the movie.

49. ARCHAEOLOGY Archaeologists found whale bones at coordinates (0, 3), (4, 7), and (5, 9). If the units of the coordinates are meters, find the area of the triangle formed by these finds.
Use Cramer’s Rule to solve each system of equations.

50. \[ \begin{align*}
6a - 7b &= -55 \\
2a + 4b - 3c &= 35 \\
-5a - 3b + 7c &= -37 
\end{align*} \]
51. \[ \begin{align*}
3a - 5b - 9c &= 17 \\
4a - 3c &= 31 \\
-5a - 4b - 2c &= -42 
\end{align*} \]
52. \[ \begin{align*}
4x - 5y &= -2 \\
7x + 3z &= -47 \\
8y - 5z &= -63 
\end{align*} \]
53. \[ \begin{align*}
7x + 8y + 9z &= -149 \\
-6x + y - 5z &= 54 \\
4x + 5y - 2z &= -44 
\end{align*} \]

54. **GARDENING** Rob wants to build a triangular flower garden. To plan out his garden he uses a coordinate grid where each of the squares represents one square foot. The coordinates for the vertices of his garden are \((-1, 7), (2, 6), \) and \((4, -3)\). Find the area of the garden.

55. **FINANCIAL LITERACY** A vendor sells small drinks for $1.15, medium drinks for $1.75, and large drinks for $2.25. During a week in which he sold twice as many small drinks as medium drinks, his total sales were $2,238.75 for 1385 drinks.
   a. Use Cramer’s Rule to determine how many of each drink were sold.
   b. The vendor decided to increase the price for small drinks to $1.25 the next week. The next week, he sold 140 fewer small drinks, 125 more medium drinks, and 35 more large drinks. Calculate his sales for that week.
   c. Was raising the price of the small drink a good business move for the vendor? Explain your reasoning.

**H.O.T. Problems** Use Higher-Order Thinking Skills

56. **REASONING** Some systems of equations cannot be solved by using Cramer’s Rule.
   a. Find the value of \[ \begin{vmatrix} a & b \\ f & g \end{vmatrix} \]. When is the value 0?
   b. Choose values for \(a, b, f,\) and \(g\) to make the determinant of the coefficient matrix 0. What type of system is formed?

57. **REASONING** What can you determine about the solution of a system of linear equations if the determinant of the coefficients is 0?

58. **ERROR ANALYSIS** James and Amber are finding the value of \[ \begin{vmatrix} 8 & 3 \\ -5 & 2 \end{vmatrix} \].

   James
   \[ \begin{vmatrix} 8 & 3 \\ -5 & 2 \end{vmatrix} = 16 - (-15) = 31 \]

   Amber
   \[ \begin{vmatrix} 8 & 3 \\ -5 & 2 \end{vmatrix} = 16 - 15 = 1 \]

   Is either of them correct? Explain your reasoning.

59. **CHALLENGE** Find the determinant of a \(3 \times 3\) matrix defined by
   \[ a_{mn} = \begin{cases} 0 & \text{if } m + n \text{ is even} \\ m + n & \text{if } m + n \text{ is odd} \end{cases} \]

60. **OPEN ENDED** Write a \(2 \times 2\) matrix with each of the following characteristics.
   a. The determinant equals 0.
   b. The determinant equals 25.
   c. The elements are all negative numbers and the determinant equals –32.

61. **WRITING IN MATH** Describe the possible graphical representations of a \(2 \times 2\) system of linear equations if the determinant of the matrix of coefficients is 0.
62. Tyler paid $25.25 to play three games of miniature golf and two rides on go-karts. Brent paid $25.75 for four games of miniature golf and one ride on the go-karts. How much does one game of miniature golf cost?

A $4.25  
B $4.75  
C $5.25  
D $5.75

63. Use the table to determine the expression that best represents the number of faces of any prism having a base with \( n \) sides.

<table>
<thead>
<tr>
<th>Base</th>
<th>Sides of Base</th>
<th>Faces of Prisms</th>
</tr>
</thead>
<tbody>
<tr>
<td>triangle</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>quadrilateral</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>pentagon</td>
<td>5</td>
<td>7</td>
</tr>
<tr>
<td>hexagon</td>
<td>6</td>
<td>8</td>
</tr>
<tr>
<td>heptagon</td>
<td>7</td>
<td>9</td>
</tr>
<tr>
<td>octagon</td>
<td>8</td>
<td>10</td>
</tr>
</tbody>
</table>

F \( 2(n - 1) \)  
G \( 2(n + 1) \)  
H \( n + 2 \)  
J \( 2n \)

64. SHORT RESPONSE  A right circular cone has radius 4 inches and height 6 inches.

What is the lateral area of the cone? (lateral area of cone = \( \pi r\ell \), where \( \ell \) = slant height)

65. SAT/ACT  Find the area of \( \triangle ABC \).

A 10 units\(^2\)  
B 12 units\(^2\)  
C 13 units\(^2\)  
D 14 units\(^2\)  
E 16 units\(^2\)

66. SHORT RESPONSE  A right circular cone has radius 4 inches and height 6 inches.

What is the lateral area of the cone? (lateral area of cone = \( \pi r\ell \), where \( \ell \) = slant height)

67. SAT/ACT  Find the area of \( \triangle ABC \).

A 10 units\(^2\)  
B 12 units\(^2\)  
C 13 units\(^2\)  
D 14 units\(^2\)  
E 16 units\(^2\)

68. **Spiral Review**

Determine the coordinates of the vertices of the image of each figure after a reflection in the given axis of symmetry. Then graph the preimage and image.  (Lesson 4-4)

66. Rectangle \( ABCD \) with vertices \( A(-1, 6), B(1, 5), C(-2, -1), \) and \( D(-4, 0); y\)-axis

67. \( \triangle EFG \) with vertices \( E(-2, 5), F(5, -3), \) and \( G(-1, -6); y = x \)

Determine whether each matrix product is defined. If so, state the dimensions of the product.  (Lesson 4-3)

68. \( A_{4 \times 2} \cdot B_{2 \times 6} \)  
69. \( C_{5 \times 4} \cdot D_{5 \times 3} \)  
70. \( E_{2 \times 7} \cdot F_{7 \times 1} \)

Graph each function.  (Lesson 2-6)

71. \( f(x) = 2|x - 3| - 4 \)  
72. \( f(x) = -3|2x| + 4 \)  
73. \( f(x) = |3x - 1| + 2 \)

69. **Skills Review**

Solve each system of equations.  (Lesson 3-2)

74. \( 2x - 5y = -26 \)  
75. \( 4y + 6x = 10 \)  
76. \( -3x - 2y = 17 \)  
77. \( 5x + 3y = -34 \)  
78. \( 2x - 7y = 22 \)  
79. \( -4x + 5y = -8 \)
Inverse Matrices and Systems of Equations

Then

• You solved systems of linear equations algebraically.
  (Lesson 3-2)

Now

1 Find the inverse of a $2 \times 2$ matrix.

2 Write and solve matrix equations for a system of equations.

Why?

• Maria’s Sandwich Shop offers three lunch options as shown at the right.

   To determine how much each individual item costs, you can solve the following matrix equation in which $w$ represents the cost of a sandwich, $s$ the cost of a side, and $d$ the cost of a drink.

   \[
   \begin{bmatrix}
   1 & 2 & 0 \\
   2 & 2 & 2 \\
   4 & 3 & 4
   \end{bmatrix}
   \begin{bmatrix}
   w \\
   s \\
   d
   \end{bmatrix}
   =
   \begin{bmatrix}
   9 \\
   16.50 \\
   30.75
   \end{bmatrix}
   \]

1 Identity and Inverse Matrices

Recall that in real numbers, two numbers are multiplicative inverses if their product is the multiplicative identity, 1. Similarly, for matrices, the **identity matrix** is a square matrix that, when multiplied by another matrix, equals that same matrix.

- **$2 \times 2$ Identity Matrix**

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

- **$3 \times 3$ Identity Matrix**

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

2 New Vocabulary

- identity matrix
- inverse matrix
- matrix equation
- variable matrix
- constant matrix

Tennessee Curriculum Standards

CLE 3103.3.3 Analyze and apply various methods to solve equations, absolute values, inequalities, and systems of equations over complex numbers.

✓ 3103.3.8 Solve a three by three system of linear equations algebraically and by using inverse matrices and determinants with and without technology.

1. **Identity Matrix for Multiplication**

   **Words**
   
   The identity matrix for multiplication is a square matrix with 1 for every element of the main diagonal, from upper left to lower right, and 0 in all other positions.
   
   For any square matrix $A$ of the same dimension as $I$, $A \cdot I = I \cdot A = A$.

   **Symbols**
   
   If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ such that
   
   \[
   \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.
   \]

Two $n \times n$ matrices are **inverses** of each other if their product is the identity matrix. If matrix $A$ has an inverse symbolized by $A^{-1}$, then $A \cdot A^{-1} = A^{-1} \cdot A = I$. 
**Example 1 Verify Inverse Matrices**

Determine whether the matrices in each pair are inverses.

**a.** \( A = \begin{bmatrix} -4 & 2 \\ -2 & 1 \end{bmatrix} \) and \( B = \begin{bmatrix} 1/4 & 1/2 \\ 1/2 & -1 \end{bmatrix} \)

If \( A \) and \( B \) are inverses, then \( A \cdot B = B \cdot A = I \).

\[
A \cdot B = \begin{bmatrix} -4 & 2 \\ -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1/4 & 1/2 \\ 1/2 & -1 \end{bmatrix} = \begin{bmatrix} -1 + 1 & 2 - 2 \\ -1/2 + 1/2 & 1 - 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\]

Since \( A \cdot B \neq I \), they are *not* inverses.

**b.** \( F = \begin{bmatrix} 3 & -5 \\ -2 & 6 \end{bmatrix} \) and \( G = \begin{bmatrix} 3/4 & 5/8 \\ 1/4 & 3/8 \end{bmatrix} \)

If \( F \) and \( G \) are inverses, then \( F \cdot G = G \cdot F = I \).

\[
F \cdot G = \begin{bmatrix} 3 & -5 \\ -2 & 6 \end{bmatrix} \cdot \begin{bmatrix} 3/4 & 5/8 \\ 1/4 & 3/8 \end{bmatrix} = \begin{bmatrix} 9/4 - 5/4 + 30/4 & 15/8 - 15/8 + 30/8 \\ -6/4 + 6/4 & 10/8 + 18/8 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

\[
G \cdot F = \begin{bmatrix} 3/4 & 5/8 \\ 1/4 & 3/8 \end{bmatrix} \cdot \begin{bmatrix} 3 & -5 \\ -2 & 6 \end{bmatrix} = \begin{bmatrix} 9/4 - 10/4 + 30/4 & -15/4 + 30/4 \\ 3/4 - 6/4 + 30/4 & -5/4 + 18/8 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

Since \( F \cdot G = G \cdot F = I \), \( F \) and \( G \) are inverses.

**Guided Practice**

1. Determine whether \( X = \begin{bmatrix} 4 & -1 \\ 2 & -2 \end{bmatrix} \) and \( Y = \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & 2/3 \end{bmatrix} \) are inverses of each other.

Some matrices do not have inverses. You can determine whether a matrix has an inverse by using the determinant.

**Key Concept. Inverse of a 2 \( \times \) 2 Matrix**

The inverse of matrix \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) is \( A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \), where \( ad - bc \neq 0 \).

Notice that \( ad - bc \) is the value of \( \text{det} \ A \). Therefore, if the value of the determinant of a matrix is 0, the matrix cannot have an inverse.
Example 2 Find the Inverse of a Matrix

Find the inverse of each matrix, if it exists.

a. \( P = \begin{bmatrix} 7 & -5 \\ 2 & -1 \end{bmatrix} \)

\[
\begin{vmatrix} 7 & -5 \\ 2 & -1 \end{vmatrix} = -7 - (-10) = 3
\]

Find the determinant.

Since the determinant does not equal 0, \( P^{-1} \) exists.

\[
P^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}
\]

Definition of inverse

\[
= \frac{1}{7(-1) - (-5)(2)} \begin{bmatrix} -1 & 5 \\ -2 & 7 \end{bmatrix}
\]

\[
a = 7, \quad b = -5, \quad c = 2, \quad d = -1
\]

\[
= \frac{1}{3} \begin{bmatrix} -1 & 5 \\ -2 & 7 \end{bmatrix}
\]

or

\[
= \frac{1}{3} \begin{bmatrix} -1 & 5 \\ -2 & 7 \end{bmatrix}
\]

Simplify.

CHECK Find the product of the matrices. If the product is \( I \), then they are inverses.

\[
\begin{bmatrix} 7 & -5 \\ 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 5 \\ -2 & 7 \end{bmatrix} = \begin{bmatrix} -7 + 10 & 35 - 35 \\ -2 + 2 & 10 - 7 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}
\]

or

\[
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

✓

b. \( Q = \begin{bmatrix} -8 & -6 \\ 12 & 9 \end{bmatrix} \)

\[
\begin{vmatrix} -8 & -6 \\ 12 & 9 \end{vmatrix} = -72 - (-72) = 0
\]

Find the determinant.

Since the determinant equals 0, \( Q^{-1} \) does not exist.

Guided Practice

2A. \( \begin{bmatrix} 3 & 7 \\ 1 & -4 \end{bmatrix} \)

2B. \( \begin{bmatrix} 2 & 1 \\ -4 & 3 \end{bmatrix} \)

Matrix Equations Matrices can be used to represent and solve systems of equations. Consider the system of equations below. You can write a matrix equation to solve this system.

\[
x + 2y = 9 \\
3x - 6y = 3
\]

\[
\rightarrow \begin{bmatrix} x + 2y \\ 3x - 6y \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \end{bmatrix}
\]

Write the left side of the matrix equation as the product of the coefficient matrix and the variable matrix. Write the right side as a constant matrix.

\[
\begin{bmatrix} 1 & 2 \\ 3 & -6 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \end{bmatrix}
\]
Then solve the matrix equation in the same way that you would solve any other equation.

\[
ax = b \\
\left(\frac{1}{a}\right)ax = \left(\frac{1}{a}\right)b \\
1x = \frac{b}{a} \\
x = \frac{b}{a}
\]

Write the equation. Multiply each side by the inverse of the coefficient, if it exists. \(A^{-1}AX = A^{-1}B\) \(IX = A^{-1}B\) \(X = A^{-1}B\)

Notice that the solution of the matrix equation is the product of the inverse of the coefficient matrix and the constant matrix.

### Real-World Example 3: Solve a System of Equations

**TRAVEL** Helena stopped for gasoline twice during a car trip. The price of gasoline at the first station where she stopped was $3.75 per gallon. At the second station, the price was $3.50 per gallon. Helena bought a total of 24.2 gallons of gasoline and spent $88.05. How much gasoline did Helena buy at each gas station?

A system of equations to represent the situation is as follows.

\[
x + y = 24.2 \\
3.75x + 3.50y = 88.05
\]

The matrix equation is

\[
\begin{bmatrix}
1 & 1 \\
3.75 & 3.50
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
=
\begin{bmatrix}
24.2 \\
88.05
\end{bmatrix}
\]

**Step 1** Find the inverse of the coefficient matrix.

\[
A^{-1} = \frac{1}{3.50 - 3.75}
\begin{bmatrix}
3.50 & -1 \\
-3.75 & 1
\end{bmatrix}
\text{or}
\frac{1}{-0.25}
\begin{bmatrix}
3.50 & -1 \\
-3.75 & 1
\end{bmatrix}
\]

**Step 2** Multiply each side of the matrix equation by the inverse matrix.

\[
\frac{1}{-0.25}
\begin{bmatrix}
3.50 & -1 \\
-3.75 & 1
\end{bmatrix}
\begin{bmatrix}
1 \\
3.75
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
=
\frac{1}{-0.25}
\begin{bmatrix}
3.50 & -1 \\
-3.75 & 1
\end{bmatrix}
\begin{bmatrix}
24.2 \\
88.05
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
=
\begin{bmatrix}
13.4 \\
10.8
\end{bmatrix}
\]

The solution is (13.4, 10.8), where \(x\) represents the amount of gas Helena purchased at the first gas station, and \(y\) represents the amount purchased at the second gas station.

**CHECK** You can check your answer by using inverses .

Enter \(\begin{bmatrix}
1 & 1 \\
3.75 & 3.50
\end{bmatrix}\) as matrix \(A\).

Enter \(\begin{bmatrix}
24.2 \\
88.05
\end{bmatrix}\) as matrix \(B\).

Multiply the inverse of \(A\) by \(B\).

### Guided Practice

3. **COMIC BOOKS** Dante and Erica just returned from a comic book store that sells new and used comics. Dante spent $11.25 on 3 new and 4 old books, and Erica spent $15.75 on 10 used and 3 new ones. If comics of one type are sold at the same price, what is the price in dollars of a new comic book?
Example 1  Determine whether the matrices in each pair are inverses.

1. \( A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \), \( B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \)

2. \( C = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \), \( D = \begin{bmatrix} 2 & 1 \\ 5 & -3 \end{bmatrix} \)

3. \( F = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \), \( G = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \)

4. \( H = \begin{bmatrix} -3 & -1 \\ -4 & -2 \end{bmatrix} \), \( J = \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix} \)

Example 2  Find the inverse of each matrix, if it exists.

5. \( \begin{bmatrix} 6 & -3 \\ -1 & 0 \end{bmatrix} \)

6. \( \begin{bmatrix} 2 & -4 \\ -3 & 0 \end{bmatrix} \)

7. \( \begin{bmatrix} -3 & 0 \\ 5 & 2 \end{bmatrix} \)

8. \( \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \)

Example 3  Use a matrix equation to solve each system of equations.

9. \(-2x + y = 9 \quad x + y = 3\)

10. \(4x - 2y = 22 \quad 6x + 9y = -3\)

11. \(-2x + y = -4 \quad 3x + y = 1\)

12. **MONEY**  Kevin had 25 quarters and dimes. The total value of all the coins was $4.

How many quarters and dimes did Kevin have?

### Practice and Problem Solving

**Extra Practice begins on page 947.**

**Example 1**  Determine whether each pair of matrices are inverses of each other.

13. \( K = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \), \( L = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \)

14. \( M = \begin{bmatrix} 0 & 2 \\ 4 & 5 \end{bmatrix} \), \( N = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \)

15. \( P = \begin{bmatrix} 4 & 0 \\ 3 & 0 \end{bmatrix} \), \( Q = \begin{bmatrix} -1 & -1 \\ \frac{2}{3} & 5 \end{bmatrix} \)

16. \( R = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix} \), \( S = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix} \)

**Example 2**  Find the inverse of each matrix, if it exists.

17. \( \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \)

18. \( \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} \)

19. \( \begin{bmatrix} 3 & 0 \\ 5 & 1 \end{bmatrix} \)

20. \( \begin{bmatrix} 1 & -1 \\ -6 & -1 \end{bmatrix} \)

21. \( \begin{bmatrix} -5 & -4 \\ 4 & 2 \end{bmatrix} \)

22. \( \begin{bmatrix} -5 & 9 \\ 4 & -8 \end{bmatrix} \)

23. \( \begin{bmatrix} 6 & -5 \\ 4 & 9 \end{bmatrix} \)

24. \( \begin{bmatrix} -4 & -2 \\ 7 & 8 \end{bmatrix} \)

25. \( \begin{bmatrix} -6 & 8 \\ 8 & -7 \end{bmatrix} \)

**Example 3**  **BAKING**  Peggy is preparing a colored frosting for a cake. For the right shade of purple, she needs 25 milliliters of a 44% concentration food coloring. The store has a 25% red and a 50% blue concentration of food coloring. How many milliliters each of blue food coloring and red food coloring should be mixed to make the necessary amount of purple food coloring?

Use a matrix equation to solve each system of equations.

27. \(-x + y = 4 \quad -x + y = -4\)

28. \(-x + y = 3 \quad -2x + y = 6\)

29. \(x + y = 4 \quad -4x + y = 9\)

30. \(3x + y = 3 \quad 5x + 3y = 6\)

31. \(y - x = 5 \quad 2y - 2x = 8\)

32. \(4x + 2y = 6 \quad 6x - 3y = 9\)

33. \(1.6y - 0.2x = 1 \quad 0.4y - 0.1x = 0.5\)

34. \(4y - x = -2 \quad 3y - x = 6\)

35. \(2y - 4x = 3 \quad 4x - 3y = -6\)
36. **POPULATIONS** The diagram shows the annual percent migration between a city and its suburbs.

[Diagram showing city and suburbs with percentages of migration]

a. Write a matrix to represent the transitions in city population and suburb population.

b. There are currently 16,275 people living in the city and 17,552 people living in the suburbs. Assuming that the trends continue, predict the number of people who will live in the suburbs next year.

c. Use the inverse of the matrix from part b to find the number of people who lived in the city last year.

37. **MUSIC** The diagram shows the trends in digital audio player and portable CD player ownership over the past five years for Central City. Every person in Central City has either a digital audio player or a portable CD player. Central City has a stable population of 25,000 people, of whom 17,252 own digital audio players and 7748 own portable CD players.

[Diagram showing digital audio player and portable CD player statistics]

a. Write a matrix to represent the transitions in player ownership.

b. Assume that the trends continue. Predict the number of people who will own digital audio players next year.

c. Use the inverse of the matrix from part b to find the number of people who owned digital audio players last year.

### H.O.T. Problems Use Higher-Order Thinking Skills

38. **ERROR ANALYSIS** Cody and Megan are setting up matrix equations for the system $5x + 7y = 19$ and $3y + 4x = 10$. Is either of them correct? Explain your reasoning.

Cody: $\begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 19 \\ 10 \end{bmatrix}$

Megan: $\begin{bmatrix} 5 & 7 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 19 \\ 10 \end{bmatrix}$

39. **CHALLENGE** Describe what a matrix equation with infinite solutions looks like.

40. **REASONING** Determine whether the following statement is always, sometimes, or never true. Explain your reasoning.

   A square matrix has a multiplicative inverse.

41. **OPEN ENDED** Write a matrix equation that does not have a solution.

42. **WRITING IN MATH** Explain how matrix equations can be used to solve systems of equations.
43. The Yogurt Shoppe sells cones in three sizes: small, $0.89; medium, $1.19; and large, $1.39. One day Santos sold 52 cones. He sold seven more medium cones than small cones. If he sold $58.98 in cones, how many medium cones did he sell?

A 11  B 17  C 24  D 36

44. The chart shows an expression evaluated for different values of \(x\).

\[
\begin{array}{c|c|c}
\text{x} & \text{x}^2 + x + 1 \\
1 & 3 \\
2 & 7 \\
3 & 13 \\
4 & 21 \\
\end{array}
\]

A student concludes that for all values of \(x\), \(x^2 + x + 1\) produces a prime number. Which value of \(x\) serves as a counterexample to prove this conclusion false?

F -4  G -3  H 2  J 4

45. SHORT RESPONSE What is the solution of the system of equations 
\[ 6a + 8b = 5 \] and 
\[ 10a - 12b = 2 \]?

46. SAT/ACT Each year at Capital High School the students vote to choose the theme of the homecoming dance. The theme “A Night Under the Stars” received 225 votes, and “The Time of My Life” received 480 votes. If 40% of girls voted for “A Night Under the Stars” and 75% of boys voted for “The Time of My Life,” how many girls and boys voted?

A 176 boys and 351 girls  B 395 boys and 310 girls  C 380 boys and 325 girls  D 705 boys and 325 girls  E 854 boys and 176 girls

50. Quadrilateral \(A'B'C'D'\) is an image after a translation of quadrilateral \(ABCD\). A table of the vertices of each rectangle is shown. (Lesson 4-4)

<table>
<thead>
<tr>
<th>Quadrilateral (ABCD)</th>
<th>Quadrilateral (A'B'C'D')</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A(2, 5))</td>
<td>(A'(-1, 13))</td>
</tr>
<tr>
<td>(B(9, 15))</td>
<td>(B'(? , ?))</td>
</tr>
<tr>
<td>(C(-14, -9))</td>
<td>(C(? , ?))</td>
</tr>
<tr>
<td>(D(-6, 1))</td>
<td>(D(-9, 9))</td>
</tr>
</tbody>
</table>

51. MILK The Yoder Family Dairy produces at most 200 gallons of skim and whole milk each day for delivery to large bakeries and restaurants. Regular customers require at least 15 gallons of skim and 21 gallons of whole milk each day. If the profit on a gallon of skim milk is $0.82 and the profit on a gallon of whole milk is $0.75, how many gallons of each type of milk should the dairy produce each day to maximize profits? (Lesson 3-4)

52. Identify the type of the function represented by each group. (Lesson 2-7)
Using a TI-83/84 Plus graphing calculator, you can solve a system of linear equations using the MATRIX function. An **augmented matrix** contains the coefficient matrix with an extra column containing the constant terms. You can use a graphing calculator to reduce the augmented matrix so that the solution of the system of equations can be easily determined.

**Example**

Write an augmented matrix for the following system of equations. Then solve the system by using a graphing calculator.

\[
\begin{align*}
2x + y + z &= 1 \\
3x + 2y + 3z &= 12 \\
4x + y + 2z &= -1
\end{align*}
\]

**Step 1** Write the augmented matrix and enter it into a calculator.

The augmented matrix \( B = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 3 & 2 & 3 & 12 \\ 4 & 1 & 2 & -1 \end{bmatrix} \).

Begin by entering the matrix.

**KEYSTROKES:**

\[
\begin{align*}
2 & \text{nd} \ [\text{MATRIX}] \rightarrow \rightarrow \ \text{ENTER} \ 3 \ \text{ENTER} \ 4 \ \text{ENTER} \ 2 \ \text{ENTER} \ 1 \\
\text{ENTER} \ 1 \ \text{ENTER} \ 1 \ \text{ENTER} \ 3 \ \text{ENTER} \ 2 \ \text{ENTER} \ 3 \ \text{ENTER} \\
4 \ \text{ENTER} \ 1 \ \text{ENTER} \ 2 \ \text{ENTER} \ (-) \ 1 \ \text{ENTER}
\end{align*}
\]

**Step 2** Find the reduced row echelon form (rref) using the graphing calculator.

**KEYSTROKES:**

\[
\begin{align*}
2 & \text{nd} \ [\text{QUIT}] \ 2 & \text{nd} \ [\text{MATRIX}] \ \alpha \ [B] \ 2 & \text{nd} \ [\text{MATRIX}] \\
\text{ENTER} \ ) \ \text{ENTER}
\end{align*}
\]

Study the reduced echelon matrix. The first three columns are the same as a 3 × 3 identity matrix. The first row represents \( x = -4 \), the second row represents \( y = 3 \), and the third row represents \( z = 6 \). The solution is \((-4, 3, 6)\).

**Exercises**

Write an augmented matrix for each system of equations. Then solve with a graphing calculator.

1. \(3x + 2y = -4\)  
   \(4x + 7y = 13\)
2. \(2x + y = 6\)  
   \(6x - 2y = 0\)
3. \(2x + 2y = -4\)  
   \(7x + 3y = 10\)
4. \(4x + 6y = 0\)  
   \(8x - 2y = 7\)
5. \(6x - 4y + 2z = -4\)  
   \(2x - 2y + 6z = 10\)  
   \(2x + 2y + 2z = -2\)
6. \(5x - 5y + 5z = 10\)  
   \(5x - 5z = 5\)  
   \(5y + 10z = 0\)
Study Guide

Key Concepts

Matrices (Lesson 4-1)
- A matrix is a rectangular array of variables or constants in horizontal rows and vertical columns.
- Equal matrices have the same dimensions and their corresponding elements are equal.

Operations (Lessons 4-2 and 4-3)
- Matrices can be added or subtracted if they have the same dimensions. Add or subtract corresponding elements.
- To multiply a matrix by a scalar \( k \), multiply each element in the matrix by \( k \).
- Two matrices can be multiplied if and only if the number of columns in the first matrix is equal to the number of rows in the second matrix.

Transformations (Lesson 4-4)
- Use matrix addition and a translation matrix to find the coordinates of a translated figure.
- Use scalar multiplication to perform dilations.
- To rotate a figure counterclockwise about the origin, multiply the vertex matrix on the left by a rotation matrix.

Identity and Inverse Matrices (Lesson 4-6)
- An identity matrix is a square matrix with ones on the diagonal and zeros in the other positions.
- Two matrices are inverses of each other if their product is the identity matrix.

Matrix Equations (Lesson 4-6)
- To solve a matrix equation, find the inverse of the coefficient matrix. Then multiply each side of the equation by the inverse matrix.

Key Vocabulary

- coefficient matrix (p. 223)
- column matrix (p. 186)
- constant matrix (p. 231)
- Cramer’s rule (p. 223)
- determinant (p. 220)
- diagonal rule (p. 221)
- dimension (p. 185)
- element (p. 185)
- equal matrices (p. 186)
- identity matrix (p. 229)
- image (p. 209)
- inverse matrix (p. 229)
- matrix (p. 185)
- matrix equation (p. 231)
- preimage (p. 209)
- rotation (p. 212)
- row matrix (p. 186)
- scalar (p. 194)
- scalar multiplication (p. 194)
- second-order determinant (p. 220)
- square matrix (p. 186)
- third-order determinant (p. 221)
- variable matrix (p. 231)
- vertex matrix (p. 209)
- zero matrix (p. 186)

Vocabulary Check

Choose the correct term from the list above to complete each sentence.

1. A(n) ___________ is a rectangular array of constants or variables.

2. A matrix can be multiplied by a constant called a(n) ___________.

3. A matrix that contains the constants in a system of equations is called a(n) ___________.

4. Each value in a matrix is called a(n) ___________.

5. The ___________ of a matrix with 4 rows and 3 columns are \( 4 \times 3 \).

6. A(n) ___________ occurs when a figure is moved about a center point, usually the origin.

7. The ___________ matrix is a square matrix that, when multiplied by another matrix, equals that same matrix.

8. In a(n) ___________ matrix, every element is zero.

9. The ___________ of \[
\begin{bmatrix}
-1 & 2 \\
2 & -3 \\
\end{bmatrix}
\] is \(-1\).

10. If the product of two matrices is the identity matrix, they are ___________.

Foldables Study Organizer

Be sure the Key Concepts are noted in your Foldable.
Lesson-by-Lesson Review

4-1 Introduction to Matrices (pp. 185–191)

11. SALES  Three competing retail stores recorded the number and type of customers that purchased items at their stores one day. The following table shows the numbers.

<table>
<thead>
<tr>
<th>Type of Customer</th>
<th>Store A</th>
<th>Store B</th>
<th>Store C</th>
</tr>
</thead>
<tbody>
<tr>
<td>adult (18 and older)</td>
<td>64</td>
<td>108</td>
<td>31</td>
</tr>
<tr>
<td>student (under 18)</td>
<td>42</td>
<td>9</td>
<td>68</td>
</tr>
</tbody>
</table>

a. Write a matrix for the numbers of customers.
b. What are the dimensions of the matrix?
c. What value is $a_{23}$?
d. What value is $a_{11}$?
e. Add the elements in columns 1 and 2 and interpret the results.
f. Would finding the sum of the rows provide any meaningful data? Explain.

Example 1

A movie house has three theatres; each theatre shows a different movie. The number of people who attended each movie is shown.

<table>
<thead>
<tr>
<th>Type of Movie</th>
<th>Theatre 1</th>
<th>Theatre 2</th>
<th>Theatre 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>matinee</td>
<td>37</td>
<td>19</td>
<td>26</td>
</tr>
<tr>
<td>evening</td>
<td>69</td>
<td>58</td>
<td>75</td>
</tr>
</tbody>
</table>

a. Write a matrix for the number of customers.
b. What are the dimensions of the matrix? $2 \times 3$
c. Add the elements in rows 1 and 2, and interpret the results.
The sum of row 1 is 82, which is the total number of customers at the matinee. The sum of row 2 is 202, which is the total number of customers at the evening show.

4-2 Operations with Matrices (pp. 193–199)

12. \[
\begin{bmatrix}
2 & -3 \\
-6 & 0
\end{bmatrix}
- \begin{bmatrix}
3 & 2 \\
-3 & 1
\end{bmatrix}
+ \begin{bmatrix}
6 & 0
\end{bmatrix}
\]

13. \[
3 \begin{bmatrix}
-2 & 0 \\
6 & 8
\end{bmatrix}
+ \begin{bmatrix}
1 & 9 \\
-3 & -4
\end{bmatrix}
\]

14. RETAIL  Current Fashions buys shirts, jeans, and shoes from a manufacturer, marks them up, and then sells them. The table shows the purchase price and the selling price.

<table>
<thead>
<tr>
<th>Item</th>
<th>Purchase Price</th>
<th>Selling Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>shirts</td>
<td>$15</td>
<td>$35</td>
</tr>
<tr>
<td>jeans</td>
<td>$25</td>
<td>$55</td>
</tr>
<tr>
<td>shoes</td>
<td>$30</td>
<td>$85</td>
</tr>
</tbody>
</table>

a. Write a matrix for the purchase price.
b. Write a matrix for the selling price.
c. Use matrix operations to find the profit on 1 shirt, 1 pair of jeans, and 1 pair of shoes.

Example 2

Find $2B + 3A$ if $A = \begin{bmatrix}
9 & 1 \\
1 & 2
\end{bmatrix}$ and $B = \begin{bmatrix}
1 & 4 \\
3 & 7
\end{bmatrix}$.

$2B = \begin{bmatrix}
2 & 8 \\
6 & 14
\end{bmatrix}$ or $\begin{bmatrix}
2 & 8 \\
6 & 14
\end{bmatrix}$

$3A = \begin{bmatrix}
9 & 1 \\
3 & 6
\end{bmatrix}$ or $\begin{bmatrix}
9 & 1 \\
3 & 6
\end{bmatrix}$

$2B + 3A = \begin{bmatrix}
2 & 8 \\
6 & 14
\end{bmatrix}
+ \begin{bmatrix}
27 & 3 \\
9 & 11
\end{bmatrix}$ or $\begin{bmatrix}
29 & 11 \\
15 & 25
\end{bmatrix}$
4–3 Multiplying Matrices (pp. 200–207)

Find each product, if possible.

15. \[
\begin{bmatrix}
3 & -7
\end{bmatrix} \cdot \begin{bmatrix}
9
\end{bmatrix}
\]

16. \[
\begin{bmatrix}
-3 & 0 & 2
\end{bmatrix}
\cdot \begin{bmatrix}
0 & 8 & -1
\end{bmatrix}
\]

17. \[
\begin{bmatrix}
2 & 11
0 & -3
-6 & 7
\end{bmatrix}
\cdot \begin{bmatrix}
0 & 8 & -5
12 & 0 & 9
4 & -6 & 7
\end{bmatrix}
\]

18. GROCERIES Martin bought 1 gallon of milk, 2 apples, 4 frozen dinners, and 1 box of cereal. The following matrix shows the prices for each item, respectively.

\[
\begin{bmatrix}
$2.59 & $0.49 & $5.25 & $3.99
\end{bmatrix}
\]

Use matrix multiplication to find the total amount of money Martin spent at the grocery store.

Example 3

Find \(XY\) if \(X = \begin{bmatrix} 0 & -6 \\ 3 & 5 \end{bmatrix}\) and \(Y = \begin{bmatrix} 8 \\ -1 \end{bmatrix}\).

\[
XY = \begin{bmatrix} 0 & -6 \\ 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ -1 \end{bmatrix}
\]

Write an equation.

\[
= \begin{bmatrix} 0(8) + (-6)(-1) \\ 3(8) + 5(-1) \end{bmatrix}
\]

Multiply columns by rows.

\[
= \begin{bmatrix} 6 \\ 19 \end{bmatrix}
\]

Simplify.

4–4 Transformations with Matrices (pp. 209–217)

Use \(\triangle ABC\) to find the coordinates of the image after each transformation.

19. translation 2 units left and 3 units up
20. dilation by a scale factor of 3
21. reflection in the \(x\)-axis
22. rotation of 180 degrees
23. QUILTS Carol used the pattern of a rectangle shown at the right for a quilt piece.

Carol wants to make a rectangle pattern that is twice as long and twice as wide. What will the new coordinates of the rectangle be?

Example 4

Find the coordinates of the vertices of the image of \(\triangle XYZ\) with \(X(-2, 3), Y(6, 6),\) and \(Z(8, -3)\) after a rotation of 270° counterclockwise about the origin.

\[
\begin{bmatrix}
0 & 1 \\ -1 & 0
\end{bmatrix}
\cdot \begin{bmatrix}
-2 & 6 & 8 \\ 3 & 6 & -3
\end{bmatrix}
\]

Write the ordered pairs in a vertex matrix. Then multiply by the rotation matrix.

\[
\begin{bmatrix}
3 & 6 & -3 \\ 2 & -6 & -8
\end{bmatrix}
\]

The vertices of \(\triangle X'Y'Z'\) are \(X'(3, 2), Y'(6, -6),\) and \(Z'(-3, -8)\).
4-5 Determinants and Cramer’s Rule (pp. 220–228)

Evaluate each determinant.

24. \[
\begin{vmatrix}
2 & 4 \\
7 & -3
\end{vmatrix}
\]

25. \[
\begin{vmatrix}
2 & 3 & -1 \\
0 & 2 & 4 \\
-2 & 5 & 6
\end{vmatrix}
\]

Use Cramer’s Rule to solve each system of equations.

26. \[3x - y = 0 \\
5x + 2y = 22\]

27. \[5x + 2y = 4 \\
3x + 4y + 2z = 6 \\
7x + 3y + 4z = 29\]

28. JEWELRY Alana paid $98.25 for 3 necklaces and 2 pairs of earrings. Petra paid $133.50 for 2 necklaces and 4 pairs of earrings. Use Cramer’s Rule to find out how much 1 necklace costs and how much 1 pair of earrings costs.

Example 5

Evaluate \[
\begin{vmatrix}
4 & -6 \\
2 & 5
\end{vmatrix}
\]

\[
\begin{align*}
\det \begin{vmatrix}
4 & -6 \\
2 & 5
\end{vmatrix} &= 4(5) - 2(-6) \\
&= 20 + 12 \\
&= 32
\end{align*}
\]

Example 6

Use Cramer’s Rule to solve \[2a + 6b = -1\] and \[a + 8b = 2\].

\[
\begin{align*}
a &= \begin{vmatrix}
1 & 6 \\
2 & 8
\end{vmatrix} \\
b &= \begin{vmatrix}
2 & -1 \\
1 & 2
\end{vmatrix}
\end{align*}
\]

\[
\begin{align*}
a &= \frac{-8 - 12}{16 - 6} = -\frac{20}{10} = -2 \\
b &= \frac{4 + 1}{16 - 6} = \frac{5}{10} = \frac{1}{2}
\end{align*}
\]

The solution is \((-2, \frac{1}{2})\).

4-6 Inverse Matrices and Systems of Equations (pp. 229–235)

Find the inverse of each matrix, if it exists.

29. \[
\begin{vmatrix}
7 & 4 \\
3 & 2
\end{vmatrix}
\]

30. \[
\begin{vmatrix}
2 & 5 \\
-5 & -13
\end{vmatrix}
\]

31. \[
\begin{vmatrix}
6 & -3 \\
-8 & 4
\end{vmatrix}
\]

Use a matrix equation to solve each system of equations.

32. \[
\begin{vmatrix}
5 & 3 \\
3 & 2
\end{vmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}
\]

33. \[
\begin{vmatrix}
3 & -1 \\
1 & 2
\end{vmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}
\]

34. HEALTH FOOD Heath sells nuts and raisins by the pound. Sonia bought 2 pounds of nuts and 2 pounds of raisins for $23.50. Drew bought 3 pounds of nuts and 1 pound of raisins for $22.25. What is the cost of 1 pound of nuts and 1 pound of raisins?

Example 9

Solve \[
\begin{vmatrix}
2 & -5 \\
3 & -6
\end{vmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 15 \\ 36 \end{bmatrix}
\]

Step 1 Find the inverse of the coefficient matrix.

\[
A^{-1} = \frac{1}{-12 + (-15)} \begin{bmatrix}
-6 & 5 \\
-3 & 2
\end{bmatrix} = \frac{1}{3} \begin{bmatrix}
-6 & 5 \\
-3 & 2
\end{bmatrix}
\]

Step 2 Multiply each side by the inverse matrix.

\[
\begin{align*}
\frac{1}{3} \begin{bmatrix}
-6 & 5 \\
-3 & 2
\end{bmatrix} \cdot \begin{bmatrix} 2 & -5 \\ 3 & -6 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} &= \frac{1}{3} \begin{bmatrix}
-6 & 5 \\
-3 & 2
\end{bmatrix} \cdot \begin{bmatrix} 15 \\ 36 \end{bmatrix} \\
\begin{bmatrix}
1 & 0 \\ 0 & 1
\end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 15 \\ 36 \end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 15 \\ 36 \end{bmatrix}
\end{align*}
\]

The solution is \(x = 15, y = 36\).
Identify each element of matrix  

\[ A = \begin{bmatrix} 2 & 2 & 7 \\ 9 & 1 & 1 \\ 8 & 0 & 8 \end{bmatrix} \]

1. \( a_{22} \)  
2. \( a_{31} \)

Perform the indicated operations. If the matrix does not exist, write \textit{impossible}.

3. \(-3 \begin{bmatrix} 4 & 0 \\ 3 & 0 \\ -3 \end{bmatrix} + 4 \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} \)

4. \begin{bmatrix} -3 & 0 \\ 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 4 \end{bmatrix} \)

5. \[ \begin{bmatrix} 2 & 0 \\ -3 & 5 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -2 \end{bmatrix} \]

6. \[ \begin{bmatrix} -5 & 7 \\ 6 & 8 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 9 & 0 \end{bmatrix} \]

7. **FINANCIAL LITERACY** Soren sells children's educational books door to door to save for college. The following table shows the cost of a set of books and the selling price. He sold 20 sets of encyclopedias, 32 sets of science books, and 14 sets of literature books.

<table>
<thead>
<tr>
<th>Type of Book</th>
<th>Cost</th>
<th>Selling Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>Encyclopedia set</td>
<td>$35</td>
<td>$105</td>
</tr>
<tr>
<td>Science books</td>
<td>$25</td>
<td>$85</td>
</tr>
<tr>
<td>Literature books</td>
<td>$40</td>
<td>$125</td>
</tr>
</tbody>
</table>

- a. Organize the data into two matrices. Then use matrix multiplication to find the total amount that Soren paid for the books.
- b. Use matrix multiplication to find the total amount that buyers paid for the books.
- c. Use matrix operations to find how much money Soren made on his sales.

8. Find \( AB - AC \) if \( A = \begin{bmatrix} 3 & 8 \\ -3 & -4 \end{bmatrix}, B = \begin{bmatrix} -7 & 5 \\ 5 & -4 \end{bmatrix} \), and \( C = \begin{bmatrix} -4 & 7 \\ 2 & 0 \end{bmatrix} \).

9. Triangle \( ABC \) with \( A(0, 2), B(1.5, -1.5) \), and \( C(-2.5, 0) \) is dilated so that its perimeter is three times the original perimeter. Write the vertex matrix for \( \triangle ABC \). Then find the coordinates of the image after the dilation.

10. **MULTIPLE CHOICE** Triangle \( A'B'C' \) is an image of \( \triangle ABC \). A table of the vertices of each triangle is shown. What are the coordinates of \( C' \)?

<table>
<thead>
<tr>
<th>( \triangle ABC )</th>
<th>( \triangle A'B'C' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A(3, 4) )</td>
<td>( A'(5, 3) )</td>
</tr>
<tr>
<td>( B(1, -2) )</td>
<td>( B'(3, -3) )</td>
</tr>
<tr>
<td>( C(5, -4) )</td>
<td>( C'(?, ?) )</td>
</tr>
</tbody>
</table>

Triangle \( XYZ \) has vertices \( X(1, 2), Y(3, 6) \), and \( Z(-1, 4) \).

11. Find the coordinates of the vertices of a triangle that is a dilation of \( \triangle XYZ \) with a perimeter two times that of \( \triangle XYZ \).

12. Find the coordinates of the vertices of \( \triangle XYZ \) after it is rotated 90 degrees counterclockwise about the origin.

13. Use a determinant to find the area of \( \triangle XYZ \).

14. **MULTIPLE CHOICE** What is the value of

\[ \begin{bmatrix} 2 & 3 & -1 \\ 0 & 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} -2 & 5 & 6 \end{bmatrix} \?

\[ F \ -44 \]

\[ G \ -1 \]

\[ H \ 1 \]

\[ J \ 44 \]

Find the inverse of each matrix, if it exists.

15. \[ \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} \]

16. \[ \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \]

17. \[ \begin{bmatrix} 6 & 3 \\ 8 & 4 \end{bmatrix} \]

18. \[ \begin{bmatrix} -3 & -2 \\ 6 & 4 \end{bmatrix} \]

Use Cramer's Rule to solve each system of equations.

19. \( 2x - y = -9 \)  
\( x + 2y = 8 \)

20. \( x - y + 2z = 0 \)  
\( 3x + z = 11 \)  
\( -x + 2y = 0 \)

21. \( 6x + 2y + 4z = 2 \)  
\( 3x + 4y - 8z = -3 \)  
\( -3x - 6y + 12z = 5 \)
Gridded Response Questions

In addition to multiple choice, short answer, and extended response questions, you will likely encounter gridded response questions on standardized tests. For gridded response questions, you must print your answer on an answer sheet and mark in the correct circles on the grid to match your answer.

Strategies for Solving Gridded Response Questions

**Step 1**
Read the problem carefully and solve.
- Be sure your answer makes sense.
- If time permits, check your answer.

**Step 2**
Print your answer in the answer boxes.
- Print only one digit or symbol in each answer box.
- Do not write any digits or symbols outside the answer boxes.
- Answers to gridded response questions may be whole numbers, decimals, or fractions.

**Step 3**
Fill in the grid.
- Fill in only one bubble for every answer box that you have written in.
  Be sure not to fill in a bubble under a blank answer box.
- Fill in each bubble completely and clearly.

Test Practice Example

Read the problem. Identify what you need to know. Then use the information in the problem to solve.

Christopher stopped for gasoline twice during a road trip. The price of gasoline at the first gas station where he stopped was $2.96 per gallon. At the second gas station, the price was $3.15 per gallon. Christopher bought a total of 28.3 gallons of gasoline between the two stops and spent a total of $86.39. How many gallons did he buy at the first gas station?
Read the problem carefully. The problem can be solved using a system of equations. Let $x$ represent the number of gallons bought at the first gas station, and let $y$ represent the number of gallons bought at the second station. The following system models the situation.

\[
\begin{align*}
x + y &= 28.3 \\
2.96x + 3.15y &= 86.39
\end{align*}
\]

The system of equations can be solved algebraically. However, if time is a concern, it may be faster and simpler to use matrices and a calculator to solve the system.

**Solve the Problem**

Enter the coefficient and constant matrices into a graphing calculator and solve using inverses.

\[
\begin{align*}
A &= \begin{bmatrix} 1 & 2.96 \\ 3.15 & 1 \end{bmatrix}, \\
B &= \begin{bmatrix} 28.3 \\ 86.39 \end{bmatrix}
\end{align*}
\]

\[
\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}B = \begin{bmatrix} 14.5 \\ 13.8 \end{bmatrix}
\]

So, Christopher bought 14.5 gallons of gasoline at the first gas station. Carefully fill in the grid to show 14.5.

**Exercises**

Read each problem. Identify what you need to know. Then use the information in the problem to solve. Copy and complete an answer grid on your paper.

1. Heather has 23 nickels, dimes, and quarters. The total value of all the coins is $3.40. If she has twice as many quarters as dimes, how many nickels does Heather have?

2. Find the value of the determinant \[
\begin{vmatrix} -1 & 4 \\ -3 & 0 \end{vmatrix}
\]

3. Kendra is displaying eight sweaters in a store window. There are four identical red sweaters, three identical brown sweaters, and one white sweater. How many different arrangements of the eight sweaters are possible?

4. Evaluate the determinant of matrix $H$.

\[
H = \begin{bmatrix} -2 & 0 & 3 \\ -5 & -7 & -1 \\ 4 & -8 & 1 \end{bmatrix}
\]

5. Polygon $DABC$ is rotated $90^\circ$ counterclockwise and then reflected over the line $y = x$. What is the $x$-coordinate of the final image of $A$?
Multiple Choice

Read each question. Then fill in the correct answer on the answer document provided by your teacher or on a sheet of paper.

1. Matrix $L$ shows the average low temperature, in degrees Fahrenheit, each month where Terrance lives. Matrix $H$ shows the monthly average high temperature.

\[
L = \begin{bmatrix}
24.1 & 27.7 & 35.9 \\
44.1 & 53.6 & 62.2 \\
66.4 & 64.9 & 57.9 \\
46.4 & 37.3 & 28.4 \\
\end{bmatrix}
\]

\[
H = \begin{bmatrix}
39.9 & 45.2 & 55.3 \\
65.1 & 74.0 & 82.3 \\
85.9 & 84.6 & 78.1 \\
66.9 & 54.5 & 44.3 \\
\end{bmatrix}
\]

Which operation would you use to find the difference between the average high temperature and the average low temperature each month?

A \hspace{.5cm} L + H \hspace{.5cm} C \hspace{.5cm} H \times L

B \hspace{.5cm} H - L \hspace{.5cm} D \hspace{.5cm} L - H

2. Find $\begin{bmatrix} 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, if possible.

F \hspace{.5cm} [-3] \hspace{.5cm} H \hspace{.5cm} \begin{bmatrix} 8 & -4 \\ 12 & 6 \end{bmatrix}

G \hspace{.5cm} [11] \hspace{.5cm} J \hspace{.5cm} \text{undefined}

3. Which equation is equivalent to $4x - 3(2x + 7) = 5x$?

A \hspace{.5cm} -2x - 21 = 5x \hspace{.5cm} C \hspace{.5cm} -2x + 21 = 5x

B \hspace{.5cm} -2x + 7 = 5x \hspace{.5cm} D \hspace{.5cm} 6x - 7 = 5x

4. Triangle $DEF$ has vertices $D(-6, 2), E(3, 5), \text{ and } F(8, -7)$. Evaluate the determinant below to find the area of the triangle.

\[
A = \frac{1}{2} \begin{vmatrix}
-6 & 2 & 1 \\
3 & 5 & 1 \\
8 & -7 & 1 \\
\end{vmatrix}
\]

F \hspace{.5cm} 54.5 \text{ square units} \hspace{.5cm} G \hspace{.5cm} 58 \text{ square units} \hspace{.5cm} H \hspace{.5cm} 60 \text{ square units} \hspace{.5cm} J \hspace{.5cm} 61.5 \text{ square units}

5. Suppose Kendall sells apples and tomatoes at a farmer’s market. If he sold 280 items one morning and earned $65.20, how many apples did he sell?

<table>
<thead>
<tr>
<th>Item</th>
<th>Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>apple</td>
<td>$0.25</td>
</tr>
<tr>
<td>tomato</td>
<td>$0.20</td>
</tr>
</tbody>
</table>

A \hspace{.5cm} 96 \hspace{.5cm} C \hspace{.5cm} 168

B \hspace{.5cm} 126 \hspace{.5cm} D \hspace{.5cm} 184

6. What are the dimensions of $D = \begin{bmatrix} 4 & -6 \\ 9 & 2 \\ 1 & 0 \\ -3 & -5 \end{bmatrix}$?

F \hspace{.5cm} 4 \text{ by } 2 \hspace{.5cm} G \hspace{.5cm} 2 \text{ by } 4 \hspace{.5cm} H \hspace{.5cm} 4 \text{ by } 8 \hspace{.5cm} J \hspace{.5cm} 8 \text{ by } 4

7. What is the solution set of $6 - |x + 7| \leq -2$?

A \hspace{.5cm} \{x \mid -15 \leq x \leq 1\} \hspace{.5cm} B \hspace{.5cm} \{x \mid -1 \leq x \leq 3\}

C \hspace{.5cm} \{x \mid x \leq -1 \text{ or } x \geq 3\} \hspace{.5cm} D \hspace{.5cm} \{x \mid x \leq -15 \text{ or } x \geq 1\}

Test-Taking Tip

Question 2 The product of a 1-by-2 matrix and a 2-by-1 matrix is a 1-by-1 matrix. So, answer choices H and J can be eliminated.
Short Response/Gridded Response

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

8. Does matrix $B$ have an inverse? Explain why or why not.

$$B = \begin{bmatrix} 3 & -2 \\ -9 & 6 \end{bmatrix}$$

9. **GRIDDED RESPONSE** Evaluate the determinant of

$$W = \begin{bmatrix} 3 & 1 & 0 \\ 2 & 5 & -4 \\ 0 & -1 & 1 \end{bmatrix}$$

10. Valeria has 14 quarters and dimes. The total value of all the coins is $2.75. Use this information to answer each question.

   a. Let $d$ represent the number of dimes that Valeria has, and let $q$ represent the number of quarters. Write a system of equations to model the situation.

   b. Write a matrix equation that can be used to solve for $d$ and $q$.

   c. Solve your matrix equation using inverses. How many dimes and quarters does Valeria have?

11. **GRIDDED RESPONSE** Andrea is using a coordinate grid to design a new deck for her backyard. The deck is represented by the intersection of $y \leq 20$, $x \leq 16$, $y \geq 0$, $x \geq 0$ and $y \leq -x + 32$. If each unit of the coordinate grid represents 1 foot, what is the area of the deck? Express your answer in square feet.

Extended Response

Record your answers on a sheet of paper. Show your work.

12. What are the coordinates of the $x$- and $y$-intercepts of the graph of $2y = 4x + 3$?

13. Describe in your own words when two matrices can be multiplied. Describe when two matrices cannot be multiplied. Give an example.

14. Use the triangle shown on the coordinate grid to answer each question.

   a. What are the vertices of $\triangle RST$?

   b. Write the coordinates in a vertex matrix.

   c. Suppose the triangle is reflected across the line $y = x$. What matrix can you multiply by the vertex matrix to find the reflected vertices?

   d. What are the vertices of the reflected triangle?

Need Extra Help?

<table>
<thead>
<tr>
<th>If you missed Question...</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
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<tr>
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<td>2-8</td>
<td>4-6</td>
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<td>3-1</td>
<td>2-2</td>
<td>4-1</td>
<td>4-4</td>
</tr>
</tbody>
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For help with TN SPI...

- 3102.3.9
- 3103.3.8
- 3102.3.5
- 3103.3.13
- 3108.3.1
- 0806.3.6
- 3108.3.3